# Statistics for Business Students

## Using Excel and IBM SPSS Statistics

## Develop your mathematical skills

by

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# Contents

# Workbook overview

The first part of this workbook starts with the simple concept of percentage numbers and applies the rules to various business and financial calculations, such as calculating profit, VAT, discounts, etc.

The middle section of the first part is dedicated to ratios and proportions. This is then applied to some business calculations, such as foreign exchange calculations, etc.

The last and the largest section of the first part of the basic maths, recaps various basic algebra rules to handle different types of numbers, their squares and roots, as well as how to handle the brackets. From there, we progress to most elementary equations and how to handle them.

The second part of the workbook is dedicated to more advanced maths, but it almost exclusively uses Excel as a primary tool to explain the concepts. The second part starts with the concept of summations and continues with a brief refresher in various more advanced topics in mathematics. The first section of the second part proceeds with the concept of product, factorial and binomial coefficients. This is followed by brief overview of $\pi$ and *e*, i.e. the two "magic" numbers that seems to be present everywhere. After the complex numbers were introduced, a relationship between $\pi$, *e*, and *i* is also covered.

The middle section of the second part also provides a refresher in logarithms and shows how they, together with *e*, can be used for some practical calculations, such as the compound interest.

This is followed by a brief refresher in some of the most fundamental concepts from trigonometry and calculus.

The final section of part 2 provides a high level, but quite advanced refresher in matrix algebra. We cover vectors, matrices, eigenvectors, eigenvalues and all the operations related to these concepts.

# Learning objectives

On successful completion of the module, you will be able to:

- Calculate percentages and convert other numbers into percentages
- Calculate a range of financial measures, including profit and loss, discounts, value added tax, simple and compound interest
- Calculate ratios and proportions
- Understand the application of simple equations to model relationships: concept of a model, basic algebra, square and square roots, indices, standard form, simple equations, formulae.
- Plot and fit a straight-line graph to model a relationship
- Calculate sums, products and factorials of numbers
- Understand binomial coefficient
- Have appreciation for the use of $\pi$ and *e*
- Understand logarithms
- Learn how to read and handle complex numbers
- Have appreciation of the relationship between $\pi$, *e* and *i*
- Understand vector operations
- Understand the application of simple trigonometry concepts
- Understand the concept of integrals and derivatives from calculus

- Understand and be able to handle basic matrix operations
- Understand the use of eigenvalues and eigenvectors

## Introduction

We've made a simple assumption that readers have acquired a certain amount of elementary mathematical proficiency to be able to follow the textbook without any difficulty. However, often this is not the case. Either the proficiency is lost and forgotten, or in fact, it has never been acquired. This is reality. If this is the case, a reader might be severely handicapped to follow the textbook.

To address these potential shortcomings, we decided to put a chapter together that will offer a refresher to most basic maths skills. As we proceeded with this task, we realised that some more ambitious students might like to go to the next level, so we added a more advanced refresher to the first part.

The first part of the chapter provides simple set of exercises to remind readers on some of the basic rules of arithmetic and algebra. Even if users rely very much on software tools, such as Excel, they often need to create formulae to execute certain operations. Without knowing the basic mathematical rules, these formulae might be wrongly constructed, which will return incorrect results.

The readers that have no need to recap their knowledge of basic mathematical operations can comfortably skip this part of the chapter. However, if some more advanced mathematical refresher is still necessary, the readers should proceed with the second part of the chapter dedicated to a more advanced maths refresher.

The second part of the chapter starts with the concept of summations and continues with a brief refresher in various topics in mathematics. From logarithms, binomial coefficients, factorials, complex numbers, matrix algebra to elementary trigonometry and ultimately vectors, matrices and eigen values.

Most of the concepts covered in this chapter are not necessary to understand the majority of this textbook. However, they represent a general mathematical foundation that will become useful if a student wishes to go beyond the basic material.

All the topics presented in this chapter are very brief and very practical. They just offer a refresher in the mechanics of calculations and no deeper understanding is offered. It is meant to be very practical to help students understand some terminology or operations that is used in certain areas of statistics.

More advanced concepts are explained in a conventional way, and then accompanied by a solution in Excel. Often Excel offers very quick and elegant solutions, but the result still needs to be understood. This is the reason why conventional method always precedes the Excel solution.

## Why is this relevant to me?

The concepts introduced in this chapter will be essential to have a clear understanding of the material covered in the textbook. This is the fundamental reason why we put this chapter together and this is why we consider it relevant

The second part is not relevant for this textbook, but it is relevant for anyone who would like to explore more advanced concepts in statistics and go beyond this textbook. In order to do this, some more complex ideas from mathematics will be necessary, which is the reason why we put the second part of the chapter together.

The last point of relevance we would like to emphasis is the use of Excel as one of the most ubiquitous platforms in business, administration, science and industry. Excel is “packed” with numerous mathematical functions that hardly ever get used. This chapter will also enable readers to get a better understanding of what is available in Excel and how to apply it in the context of more advanced mathematical skills.

# Part 1 Basic maths skills

## 1.1 Percentages

Percentage (written %) means ‘out of one hundred’ e.g. 12% means ‘twelve out of a hundred’ or $\frac{12}{100}$. 50% means ‘50 out of a hundred’ or $\frac{50}{100}$. Fractions and decimals can easily be changed into percentages and vice-versa.

### Changing fractions to percentages

To change a fraction into a percentage, multiply the fraction by $\frac{100}{1}$ .

**Example 1.1.1** Convert $\frac{3}{5}$ to a percentage

$$\frac{3}{5} = \frac{3}{5} \times \frac{100}{1} = \frac{300}{5} = 60\%$$

The answer is 60% (said 'sixty per cent'). Remember to use your rules of fractions, and to cancel where possible.

**Example 1.1.2** Convert $1\frac{2}{5}$ to a Percentage. First change the mixed number to an improper fraction, then multiply by 100.

$$1\frac{2}{5} = \frac{(5 \times 1)+2}{5} = \frac{7}{5} \times \frac{100}{1} = \frac{700}{5} = 140\%$$

***Student Exercise X1.1.1:***

Change these fractions to percentages:

1. $\frac{4}{5}$ 2. $\frac{1}{3}$ 3. $\frac{3}{4}$ 4. $\frac{2}{7}$ 5. $1\frac{1}{5}$

### Changing decimals to percentages

To change a decimal to a percentage, multiply the decimal by 100.

**Example 1.1.3** Change 0.82 to a percentage = 0.82 x 100 = 82%
**Example 1.1.4** Change 0.175 to a percentage = 0.175 x 100 = 17.5%
**Example 1.1.5** Change 0.7 to a percentage 0.7 = 0.70 x 100 = 70% (Remember that 0.7 can be written as 0.70).
**Example 1.1.6** Change 1.67 to a percentage = 1.67 x 100 = 167%

***Student Exercise X1.1.2:***

Change these decimals to percentages:

1. 0.65   2. 0.375   3. 0.89   4. 0.6   5. 2.34

## Changing percentages to fractions

To change a percentage to a fraction, divide by 100.

**Example 1.1.7** Change 75% to a fraction

$$75\ \% = \frac{75}{100} = \frac{3}{4}$$

**Example 1.1.8** Change $12\frac{1}{2}$% to a fraction.

First change the mixed number to an improper fraction.

$$12\frac{1}{2}\% = \frac{12\frac{1}{2}}{100} = \frac{12\frac{1}{2}\times 2}{100\times 2} = \frac{25}{200} = \frac{1}{8}$$

**Example 1.1.9** Change 120% to a fraction

$$120\% = \frac{120}{100} = \frac{6}{5}$$

***Student Exercise X1.1.3:***

Convert these percentages to fractions:

1. 25%   2. 30%   3. 140%   4. 33 1/3 %   5. 37 1/2 %

## Changing percentages to decimals (decimal fractions)

To change a percentage to a decimal (decimal fraction), divide the percentage by 100.

**Example 1.1.10** Change 54% to a decimal

$$54\% = \frac{54}{100} = 0.54$$

**Example 1.1.11** Change 2.3% to a decimal fraction

$$2.3\% = \frac{2.3}{100} = 0.023$$

**Example 1.1.12** Change 32.73% to a decimal fraction

$$32.73\% = \frac{32.73}{100} = 0.3273$$

***Student Exercise X1.1.4:***

Convert these percentages to decimals:

1. 63%   2. 4.7%   3. 51.65%

## Percentage change

A number can be increased or decreased by a given percentage, e.g. shoes in a sale, may be decreased (or reduced) by 10%, you may have to pay a deposit of 20% if you are buying a video.

**Example 1.1.13** What is 20% of £50?

Remember that 'of' means multiply. 20% means 20 divided by 100, therefore 20% of £50 is now written

$$20\% \text{ of } £50 = \frac{20}{100} \times \frac{£50}{1} = £10 \text{ or } \frac{20 \times 50}{100} = £10$$

**Example 1.1.14** What is 12 1/2 % of £160?

$$12\frac{1}{2}\% \text{ of } £160 = \frac{12\frac{1}{2}}{100} \times \frac{£160}{1} = £20$$

***Student Exercise X1.1.5:***

Solve the following percentage problems:

| | | | | | |
|---|---|---|---|---|---|
| 1. | 10% of 150 | 2. | 30% o £70 | 3. | 5% of 300 |
| 4. | 12% of 30 | 5. | 12.5% of 240 | | |

## Increasing a number by a given percentage

There are several methods which can be used, two of which are shown here.

**Example 1.1.15** Increase £50 by 6%

If we find 6% of £50, this gives the actual amount of the increase. This increase must then be added to the original £50 to give the final price.

Increase = 6% of £50 = $\frac{6}{100} \times \frac{£50}{1} = £3$

The increase of £3 must now be added to £50.

New Value = Old Value + Increase = £50 + £3 = £53

The alternative is to multiply 50 by 1.06 = 50 × 1.06 = £53. As 6% is 0.06, we add this to 1 to get 1.06 that serves as the multiplier to calculate the percentage **increase**.

## Decreasing a number by a given percentage

Using the same figures as in the example above, we know that 6% of £50 is £3. However, because we are looking for a decrease, the £3 must be subtracted from £50, giving the answer of £47.

The same alternative as above, is to multiply 50 with 0.94 = 50 × 0.94 = £47. As before, 6% is 0.06, we subtract this from 1 to get 0.94 that serves as the multiplier to calculate the percentage **decrease**.

***Student Exercise X1.1.6:***

1. Increase £200 by 4%
2. Decrease £200 by 4%
3. Increase £420 by 10%
4. Decrease £420 by 10%

## Making a number a percentage of another number

**Example 1.1.16** What percentage is 43 out of 86? This is written as

$$\frac{43}{86} \times \frac{100}{1} = \frac{43 \times 100}{86} = 50\%$$

**Example 1.1.17** What percentage is 50 out of 150? This is written as

$$\frac{5}{150} \times \frac{100}{1} = \frac{1}{3} \times \frac{100}{1} = \frac{100}{3} = 33\frac{1}{3}\%$$

***Student Exercise X1.1.7:***

What percentage is?

1. 12 out of 48
2. £30 out of £150
3. £200 out of £700
4. 0.5 out of 2.5
5. 1000 out of 8000

# 1.2 Ratio and Proportion

## Ratio

A ratio gives us a way of comparing two or more quantities. However, two or more quantities can only be compared when they are in the same units.

**Example 1.2.18** Find the ratio of 2 cm to 6 cm.

First check that the units are the same (in this example they are in centimetres), then write down the numbers as shown below.

2 : 6 (No need to write the units)
2 : 6 = 1: 3 (Numbers cancelled by 2)

This is said 'one to three' and can be written as a fraction $\frac{1}{3}$. It tells us that one measurement is 3 times greater than the other (i.e. 6cms is 3 times greater than 2 cm).

**Example 1.2.19** Express the ratio 75p to £1 in its simplest form.

First change everything to pence. Ratio becomes 75:100 and can be cancelled down to 3:4. Therefore, the number is 3:4. Said 'in the ratio of three to four' and can be written as a fraction $\frac{3}{4}$.

**Example 1.2.20** Express $4 : \frac{1}{3}$ in its lowest terms.

These terms must be in the same units - we cannot have mixed numbers and fractions. So, multiply one-third by 3 to make it into a whole number = 1. Of course, the 4 has also to be multiplied by 3 giving 12 (This is in the rules of fractions if you need to revise it!). So, we now have $4 : \frac{1}{3}$ = 12 : 1.

***Student Exercise X1.2.8:***

1. Express the following ratios in their lowest terms:
   (a) 2 : 4 (b) 8 : 12 (c) 60 : 150 (d) 18 : 15 (e) 4 : 1/2
2. Express the ratio £5 to 75p in its lowest terms.
3. Express the ratio 400 m to 2 km as a fraction in its lowest terms.

## Dividing an amount into proportional parts

**Example 1.2.21** Divide £500 into two parts in the ratio 2: 3.

1. Find the total number of parts 2 : 3 = 2 + 3 = 5 parts.
2. The total number of parts is equal to the total amount of money 5 parts = £500.
3. To find 1 part, divide the total amount of money by the total number of parts
   1 part = $\frac{£500}{5}$ = £100

4. To find the value of 2 and 3 parts, multiply 1 part by 2, and then multiply 1 part by 3.

   2 parts = £100 x 2 = £200 and 3 parts = £100 x 3 = £300

   To check, add 2 and 3 parts together (£200 + £300 = £500).

The required ratio is £200: £300

**Example 1.2.22** A line 30 cm long is to be divided into 3 parts in the ratio 2:3:5. Find the length of the LONGEST part.

2: 3: 5 = 10parts

10 parts = 30 cm

$$1 \text{ part} = \frac{30\,\text{cm}}{10} = 3\,\text{cm}$$

The longest part = 5 parts

Therefore, 5 parts = 5 x 3cm = 15cm

The longest part is 15 cm.

**Example 1.2.23** Two amounts of money are in the ratio 4:3. If the first amount is £24, what is the second amount? This time we are told that the first amount of money is £24; we are not given the total.

4 parts = £24

1 part = £6

Therefore, 3 parts = £6 x 3 = £18

***<u>Student Exercise X1.2.9:</u>***

1. Divide £300 in the ratio of 2:1.
2. Divide £60 in the ratio 5:7
3. A line is divided into 3 parts 2:3:7. If the line is 84 cm long calculate the length of each part?
4. A sum of money is divided in the ratio of 2:3. If the larger amount is £18, what is (i) the other amount and (ii) the total sum of money?
5. £600 is divided amongst three children in the ratio of their ages. John 5 years old, Claire is 7 years old and Robert is 8 years old. How much money does Claire receive?

## Direct proportion

If two quantities increase or decrease at the same rate, they are said to vary in **direct proportion** to one another, which means that if 2 ice-lollies cost 24p, then we would know that 4 ice-lollies would cost 48p, 6 would cost 72p and 1 would cost 12p.

Double the amount - double the cost
Treble the amount - treble the cost
Half the amount - half the cost

**Example 1.2.24** If 3 kg of apples cost £1.20, how much will 5 kg cost?

If 3 kg = £1.20

1 kg = $\frac{£1.20}{3}$ = £0.40 (or 40p)

So, 5 kg will cost = 5 x 40p = 200p = £2.

**Example 1.2.25** A car travels 100 km in 2 hours. How long will it take to travel 250 km?

100 km in 2 hours

1 km = $\frac{2}{100}$ hours = $\frac{1}{50}$ hours

250 km = 250 $\times \frac{1}{50}$ hours = 5 hours

Therefore, for 250 km, it would take 5 hours.

***<u>Student Exercise X1.2.10:</u>***

1. 7 pears cost 84p. What is the cost of 5 pears?
2. 5 kg of potatoes cost 40p. What is the cost of 8 kg?
3. A train travels 300 km in 5 hours. How long will it take to complete a journey of 450 km?
4. Three metres of wood costs £2.25. What is the cost of 7 m?
5. Two bottles of wine fill 8 wine glasses (big glasses!). How many glasses of wine can be poured from 5 bottles?

## Foreign currency and exchange

A rate of exchange exists to convert one currency to another currency. For example, on the 15/4/2008 the exchange rates were as follows for the £, $, and €.

| £ | $ | € |
|---|---|---|
| 1 | 1.3981 | 1.1427 |

Table 1.4.1 Rate of Exchange 23/4/2018
Source: BBC – Business – Market Data web site

The direct proportion method can be used to solve many foreign exchange questions.

**Example 1.2.26** Consider the problem of converting £2500 into Euros?

We note that the conversion rate is £1 = €1.2428.

Therefore, by direct proportion £2500 = 2500 × £1 = 2500 × €1.427 = €2856.75.

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | | | | | |
| 2 | | Conversion rates: | £ | $ | € |
| 3 | | | 1 | 1.3981 | 1.1427 |
| 4 | | | | | |
| 5 | | English pounds £= | £ 2,500.00 | | |
| 6 | | Value of £2500 in Euros= | € 2,856.75 | | |

Figure 1.4.1 shows how easy it would be to use Excel to solve this problem.

**Example 1.2.27** Consider the problem of converting $1500 into €'s?

We note that the conversion rate is $1.3981 = €1.1427. Therefore, by direct proportion $1 = €1.1427/$1.3981.

$1500 = 1500 × $1 = 1500 × 1.1427/1.3981 = €1227.

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | | | | | | |
| 2 | | Conversion rates: | £ | $ | € | |
| 3 | | | 1 | 1.3981 | 1.1427 | |
| 4 | | | | | | 0.82 |
| 5 | | US Dollars $ = | $ 1,500 | | | |
| 6 | | Value of $1500 in Euros= | € 1,226 | | | |

Figure 1.4.2 shows how easy it would be to use Excel to solve this problem.

***<u>Student Exercise X1.2.11:</u>***

Using the conversion rates for the £, $, and €, calculate:

1. A man takes £200 to France. How many Euros does he receive in return?
2. A businessman spends 600 euro's on travelling in France. How much does he spend in £'s.

## Inverse proportion

If an increase in one quantity produces a decrease in another, then this is said to be a case of **inverse proportion**.

**Example 1.2.28** If 10 men take 8 days to build a wall, how long will it take 4 men to do the same job?

10 men take 8 days

1 man takes 10 x 8 = 80 days

So, 4 men take $\frac{80}{4}$ days = 20 days

**Example 1.2.29** 15 women can pack 2000 articles into boxes in 3 days, how long will it take 10 women to pack the same quantity? Here, the details about how many articles there were is not relevant.

15 women take 3 days

1 woman takes 3 x 15 = 45 days

10 women take = $\frac{45}{10}$ days = 4.5 days

***Student Exercise X1.2.12:***

1. 4 men can decorate a house in 3 days. How long does it take 2 men?
2. An amount of money is divided amongst 8 children. Each child receives £9. If the same amount of money was divided amongst 12 children, how much would each child receive?
3. 20 men produce 1000 articles in 5 days. How long would it take 25 men to produce the same number of articles?

# 1.3 Basic Algebra

## Directed numbers

This unit is about positive and negative numbers.

**Positive numbers**

These, you know very well. They are numbers such as:

- 3 which can be written as +3
- 46 which can be written as +46
- 14.67 which can be written as +14.67
- a which can be written as +a

Any number or letter, which is written without a sign, is a positive number. Positive numbers may contain a plus sign, but it is common to see them with no sign at all.

**Negative numbers**

These are numbers (and letters) which have a minus sign in front of them:

- Minus 3 is written -3

- Minus 46 is written -46
- Minus 14.67 is written -14.67
- Minus a is written -a

A negative number, or letter, always has a minus sign in front of it.

**Adding and subtracting directed numbers**

As you can see all numbers have a direction - positive and negative terms is best shown, at this stage, by using a number line and doing addition and subtraction along the number line.

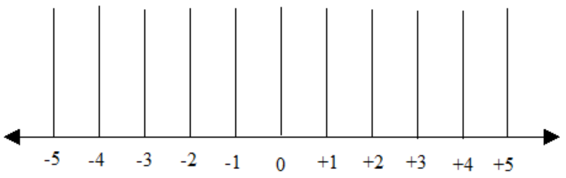


Figure 1.3.1 Number line

The number line is infinitely long, because the set of positive and negative numbers has no end. The line drawn above is just a short part of the number line.

**Addition using the number line**

**Example 1.3.30** If you start on 0 and add 3, you move 3 places to the right, answer is +3.

**Example 1.3.31** Start at +1 and add 3. Your answer is +4.

**Example 1.3.32** Start at -1 and add 3. Your answer is +2.

**Example 1.3.33**

Start at -2 and add 3. Your answer is +1. When you add, you move to the right along the number line. What do you think happens when you subtract a number? Yes! you move to the left along the number line when you subtract.

**Subtraction**

**Example 1.3.34** Start at 0 and subtract 1. Your answer is -1.

**Example 1.3.35** Start at 0 and subtract 2. Your answer is -2.

**Example 1.3.36** Start at -1 and subtract 2. Your answer is -3.

**Example 1.3.37** Start at +1 and subtract 2. Your answer is -1

Draw a number line which goes from -20 through 0 up to +20. Make sure the distance between each point is the same (one unit). Try the examples shown above, then some of your own to prove that it really works!

**Multiplication and division of directed numbers**

Multiplication rules

Positive multiplied by positive = +

(+) * (+) = +

Negative multiplied by negative = +

(-) * (-) = +

Positive multiplied by negative = -

(+) * (-) = -

Negative multiplied by positive = -

(-) * (+) = -

**Example 1.3.38**

| | |
|---|---|
| (+3) x (+4) = +12 | (-3) x (-4) = +12 |
| (+3) x (-4) = -12 | (-3) x (+4) = -12 |

Remember $(+3)^2$ and $(-3)^2$ both equal +9. Written out fully (+3)×(+3) = + 9 and (-3) × (-3) = +9. Remember ( )( ) means times, $(+10)^2$ and $(-10)^2$ = 100. Written out fully, (+10) (+10) = 100 and (-10) (-10) = 100.

***<u>Student Exercise X1.3.13:</u>***

| | | | | | |
|---|---|---|---|---|---|
| 1. | (+3) + (+9) | 2. | (+10) + (-5) | 3. | (-1 5) + (+2) |
| 4. | (-20) + (-20) | 5. | (+13) - (+10) | 6. | (+24) - (-12) |
| 7. | (-21 ) - (+21 ) | 8. | (-21 ) - (-21 ) | 9. | (+12) + (-12) - (-12) |
| 10. | (+100) - (-50) + (+20) | | | | |

Remember when you see a bracket everything inside the bracket is multiplied by the number or letter with the sign which is outside the bracket.

Division

Positive divided by positive = +

$$\frac{+10}{+5} = +2$$

Negative divided by negative = +

$$\frac{-10}{-5} = +2$$

Positive divided by negative = -

$\frac{+10}{-5} = -2$

Negative divided by positive = -

$\frac{-10}{+5} = -2$

Summary of rules:

1. When multiplying or dividing like signs, the answer will be positive.
2. When multiplying or dividing unlike signs, the answer will be negative.
3. Multiply or divide numbers as normal.

***Student Exercise X1.3.14:***

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1. | $3 \times 4$ | 2. | $-3 \times -4$ | 3. | $(-10)(-4)$ | 4. | $(+3)^2$ |
| 5. | $(-4)^2$ | 6. | $(-12)(+3)$ | 7. | $+15 \div +5$ | 8. | $-15 \div -5$ |
| 9. | $+15 \div -5$ | 10. | $+1000 \div -10$ | 11. | $+12 \div -6$ | 12. | $-36 \div +6$ |
| 13. | $+24 \div -6$ | 14. | $+16 \div -3$ | 15. | $+14 \div -3$ | 16. | $-125 \div -5$ |

## Basic algebra

In algebra, letters are used as well as numbers. Making algebraic expressions is really like making up sentences and is quite straightforward once you have learned the rules. If you are asked to make up an algebraic expression you may choose which letter(s) you wish. Look at these examples where you are asked to write down the sentences in algebraic forms.

**Example 1.3.39** Five times a number.

Let the number be d

Five times d = $5 \times d$

$5 \times d$ can be written as 5d or $5 \cdot d$

The more common way is 5d.

**Example 1.3.40** Three more than a number

Let the number be a

Three more than a = a + 3 or 3 + a

(Whichever way round, the answer is the same.)

**Example 1.3.41** Seven less than a number

Let the number be g

Seven less than g = g - 7

**Example 1.3.42** The sum of two numbers

Let the numbers be j and k

The sum of j and k = j + k

**Example 1.3.43** A number multiplied by itself

Let the number be c

c multiplied by itself = c x c = $c^2$

**Example 1.3.44** Half the number

Let the number be s

Half of $s = \frac{s}{2}$

**Example 1.3.45** The product of two numbers

(Product means multiply)

Let the numbers be y and z

The product of y and z = yz or zy

**Example 1.3.46** One number divided by three times another number

Let one number be m and the other number is n

One number divided by three times the other = $\frac{m}{3n}$

***Student Exercise X1.3.15:***

Write down the following as algebraic expressions:

1. Four times a number?
2. A quarter of a number?
3. Eight less than a number?
4. Six more than a number?
5. The sum of three numbers?
6. Three times the product of two numbers?
7. Six times a number, plus five times a second number?
8. Four times a number, minus another number?

## Substitution

When a letter is replaced by a number in an expression this is called substitution.

**Example 1.3.47** In the following 9 examples a = 2, b = 3, c = 4.

1. 5a means 5 x a = 5 x 2 = 10
2. b + c = 3 + 4 = 7
3. c - a = 4 - 2 = 2
4. 5b + 12a = 5(3) + 12(2) = 15 + 24 = 39
5. ab = 2 x 3 = 6
6. abc = 2 x 3 x 4 = 24
7. bc = 3 x 4 = 12
8. 6 - c = 6 - 4 = 2
9. 3ac = 3 x 2 x 4 = 24

***<u>Student Exercise X1.3.16:</u>***

If a = 1, b = 2, c = 3, d = 4, e = 5

Find the values of:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1. | 3 + b | 2. | 2a | 3. | c + d | 4. | e – c | 5. | 2c + 3d |
| 6. | 4a - 2b | 7. | bc | 8. | de + d | 9. | abcd | 10. | de/b |

What is different about $a^3$ and 3a? If you are in any doubt about an expression write it out in full. So $a^3 = a \times a \times a$ and $3a = 3 \times a$. These, as you know, will give different answers if the value of 'a' is known. If a = 2, then $a^3 = 2 \times 2 \times 2 = 8$ but $3a = 3 \times a = 3 \times 2 = 6$.

**Example 1.3.48** If a = 2, b = 3 and c = 5, find the values of the following.

1. $a^4 = a \times a \times a \times a = 2 \times 2 \times 2 \times 2 = 16$
2. $b^2 = b \times b = 3 \times 3 = 9$
3. $b^3 = b \times b \times b = 3 \times 3 \times 3 = 27$
4. $ac^2 = a \times c \times c = 2 \times 5 \times 5 = 50$
5. $3c = 3 \times c = 3 \times 5 = 15$
6. $4a = 4 \times a = 4 \times 2 = 8$
7. $2b^2 = 2 \times b \times b = 2 \times 3 \times 3 = 18$
8. $\frac{5a^2}{c} = \frac{5\times2\times2}{5} = 4$
9. $2c^2 + 2b^3 = 2 \times 5 \times 5 + 2 \times 3 \times 3 \times 3 = 50 + 54 = 104$

***<u>Student Exercise X1.3.17:</u>***

If p = 2, q = 3, r = 4, s = 5

Find the values of:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1. | $q^2$ | 2. | $r^3$ | 3. | $2S^2$ | 4. | $qp^2$ |
| 5. | $3p^2 + r^3$ | 6. | $2s^2 + p^3$ | 7. | $2q^2 + 3p^2$ | 8. | $rs^2$ |
| 9. | $\frac{2q^2}{r}$ | 10. | $\frac{s^2}{p}$ | | | | |

## Substitution with positive and negative numbers

**Example 1.3.49** If a = 1, b = - 2, c = 3, d = - 4, e = 5

1. $a + b = 1 + (-2) = 1 - 2 = -1$
2. $a - b = 1 - (-2) = 1 + 2 = 3$
3. $b^2 + 2e = (-2)(-2) + 2(5) = 4 + 10 = 14$
4. $ed^2 = 5(-4)(-4) = 5 \times (16) = 80$
5. $(ed)^2 = (5 \times (-4))^2 = (-20)^2 = 400$

***Student Exercise X1.3.18:***

X = -1, y = 2, z = 3

1. $2x + 3y$ 2. $xyz$ 3) $x^2y$ 4. $x + y + z^2$ 5. $(2x + y)^2$

## Addition and subtraction of algebraic terms

You can only add or subtract algebraic terms if they have the same letter(s) e.g. b's can only be added to b's, k's can only be added to k's, fg's can only be added to fg's, $g^2$'s can only be added to $g^2$'s.

**Example 1.3.50** Consider the problem 3a + 2a =? Think of it as adding 3 apples to 2 apples. Your answer would be 5 apples - in other words only the numbers are added. Therefore, 3a+2a = 5a.

**Example 1.3.51**

1. $6a - 2a = 4a$
2. $8a - 6a + 7a = 2a + 7a = 9a$
3. $2xy + 6xy = 8xy$ (remember that yx is the same as xy)
4. $3ab - ab = 2ab$ (ab is really 1ab)
5. $12xy + 5xy - 6xy = 17xy - 6xy = 11xy$
6. $a + b = a + b$
7. $3a + 2b - a + 3b = 3a - a + 2b + 3b = 2a + 5b$
8. $a^2 + 2a^2 + 3a = 3a^2 + 3a$

If there is no sign in front of the letter this is, as you know, assumed to be positive.

***Student Exercise 1.3.19:***

1. $4a + 10a$ 2. $11a - 6a$ 3. $6xy + 2xy$
4. $6a - 2a + 3a$ 5. $11b + 2b - 7b$ 6. $3a + 2b - a$
7. $8b - 6a - 2b - 7a$ 8. $3b^2 + 2b + 5b^2$ 9. $6s^2t + 3s^2t$
10. $2b^2 + 3b + 6b^2 + 4b$

## Multiplication and division of algebraic fractions

The same rules apply as for directed numbers. Read through the following examples to clarify the rules.

x times y = xy
5x times 3y = 15xy

Multiply the numbers together and then the algebraic terms.

**Example 1.3.52** a times a = $a \times a = a^2$

**Example 1.3.53** 4b times 3b times 5b = $4 \times 3 \times 5 \times b \times b \times b = 60 b^3$
**Example 1.3.54** $2y \times 5z = 2 \times 5 \times y \times z = 10 yz$
**Example 1.3.55** $a \times (-b) = - ab$
**Example 1.3.56** $(-2a) \times (5b) = - 10 ab$
**Example 1.3.57** $(2a) \times (-5b) = - 10 ab$
**Example 1.3.58** $(-2a) \times (-5b) = 10 ab$
**Example 1.3.59** $3a^2 \times 2a = 3 \times a \times a \times 2 \times a = 6a^3$
**Example 1.3.60** $\frac{4a}{2b} = \frac{2a}{b}$
**Example 1.3.61** $\frac{3x}{4y} = \frac{3x}{4y}$ (no change)
**Example 1.3.62** $\frac{4a^3}{2a} = \frac{4 \times a \times a \times a}{2 \times a} = 2a^2$
**Example 1.3.63** $\frac{12a^3bc^2}{4a^2c} = \frac{12 \times a \times a \times a \times b \times c \times c}{4 \times a \times a \times c} = 3abc$

***Student Exercise X1.3.20:***

| | | | | | |
|---|---|---|---|---|---|
| 1. | 2a * 3a | 2. | 2a * 3b | 3. | 4 * 6a |
| 4. | (- 2a) * (6a) | 5. | (2s) * (-6t) | 6. | (-2s) * (-6t) |
| 7. | $4a^2$ * $2a^2$ | 8. | $3b^2$ * $2a^2$ | 9. | (- a) * (- b) |
| 10. | $\frac{12a^2}{3a}$ | 11. | $\frac{4a^2b}{2ab^2}$ | 12. | $\frac{8a^2b^2c^2}{2abc}$ |
| 13. | $\frac{(-a)}{b}$ | 14. | $\frac{(-6x)}{(-2xy)}$ | 15. | $\frac{9a^2bc}{27a^2bc}$ |

## Brackets

Brackets are used in mathematics as a type of shorthand. When removing the brackets everything inside the brackets is multiplied by the expression outside the bracket.

**Example 1.3.64** 2(a + b) = 2a + 2b
**Example 1.3.65** 3(f + g) = 3f + 3g
**Example 1.3.66** a(j + k) = aj + ak
**Example 1.3.67** 2(a- b) = +2a - 2b
**Example 1.3.68** 4(3a - 2b) = 12a - 8b
**Example 1.3.69** 3a(5b - 6x) = 15ab- 18ax
**Example 1.3.70** $2x(3 + 2x) = 6x + 4x^2$

Look at this example - 2x(3 + x) means that+3 and +x must both be multiplied by -2x. Write it like this $(- 2x) \times (3) + (- 2x) \times (x) = - 6x - 2x^2$.

**Rule**

When a bracket has a minus sign in front of it, the signs inside the bracket are changed when the bracket is removed. Look at the following examples:

**Example 1.3.71** - 2(3 + 6a) = - 6 - 12a
**Example 1.3.72** - 3(4 - 3b) = -12 + 9b
**Example 1.3.73** - (a - b) = - a + b
**Example 1.3.74** - (a + b) = - a - b

In Example 1.3.73 and Example 1.3.74 there was only a minus sign in front of the bracket, but really this is a short way of saying that -1 is in front of the bracket. This is a very important point to remember.

***Student Exercise X1.3.21:***

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | $2(x + 3)$ | 2. | $4(a + b)$ | 3. | $6(a - b)$ | 4. | $5(x - 3)$ |
| 5. | $3(4x + 2y)$ | 6. | $-(m + n)$ | 7. | $-2(3x + 5)$ | 8. | $-3(4 - 6x)$ |
| 9. | $-(2p + 3q)$ | 10. | $4b(3a - b)$ | | | | |

## Removing brackets and simplifying

In this type of question, you have to multiply out the brackets first and then collect all the 'like' terms together.

**Example 1.3.75**

$2(X + 6) + 3(X + 5) = 2 \times X + 2 \times 6 + 3 \times X + 3 \times 5$
$2(X + 6) + 3(X + 5) = 2X + 12 + 3X + 15$
$2(X + 6) + 3(X + 5) = 5X + 27$

**Example 1.3.76**

$2(X + 3) + (X - 2) = 2 \times X + 2 \times 3 + 1 \times X - 1 \times 2$
$2(X + 3) + (X - 2) = 2X + 6 + X - 2$
$2(X + 3) + (X - 2) = 3X + 4$

**Example 1.3.77**

$X(3X + 4) - 2(X^2 - X) = X \times 3X + 4 \times X - 2 \times X^2 - 2 \times (-X)$
$X(3X + 4) - 2(X^2 - X) = 3X^2 + 4X - 2X^2 + 2X$
$X(3X + 4) - 2(X^2 - X) = X^2 + 6X$

**Example 1.3.78**

$3(a - b) - (2a - b) + 4(a - 2b) = 3a - 3b - 2a + b + 4a - 8b$
$3(a - b) - (2a - b) + 4(a - 2b) = 3a - 2a + 4a - 3b + b - 8b$
$3(a - b) - (2a - b) + 4(a - 2b) = a + 4a - 2b - 8b$
$3(a - b) - (2a - b) + 4(a - 2b) = 5a - 10b$

***Student Exercise X1.3.22:***

1. $2(x + 2) + 3(x + 4)$ 2. $3(x - 6) - 2(x - 4)$ 3. $x(2x + 1) - 4(x^2 + 1)$
4. $4(a - b) - 2(a + b) + 6(a + b)$ 5. $4x(x + 6) - 2(x^2 - 3) + 5(x^2 + x + 2)$

## Squares

The square of a number is that number multiplied by itself.

**Example 1.3.79** 'three squared' is written $3^2$ which means $3 \times 3 = 9$ and 'four squared' is written $4^2$ which means $4 \times 4 = 16$.

**Example 1.3.80** $50^2 = 50 \times 50 = 2500$
**Example 1.3.81** $(0.3)^2 = 0.3 \times 0.3 = 0.09$
**Example 1.3.82** $(0.03)^2 = 0.03 \times 0.03 = 0.0009$

## Square roots

The sign $\sqrt{\ }$ means 'the square root of', and you must find a number which when multiplied by itself gives the answer in the sign.

**Example 1.3.83** $\sqrt{16} = \sqrt{4 \times 4} = 4$
**Example 1.3.84** $\sqrt{25} = \sqrt{5 \times 5} = 5$
**Example 1.3.85** $\sqrt{100} = \sqrt{10 \times 10} = 10$
**Example 1.3.86** $\sqrt{10000} = \sqrt{100 \times 100} = 100$
**Example 1.3.87** $\sqrt{0.25} = \sqrt{\frac{25}{100}} = \frac{\sqrt{25}}{\sqrt{100}} = \frac{5}{10} = \frac{1}{2} = 0.5$
**Example 1.3.88** $\sqrt{0.0009} = \sqrt{\frac{9}{10000}} = \frac{\sqrt{9}}{\sqrt{10000}} = \frac{3}{100} = 0.03$
**Example 1.3.89** $\sqrt{0.16} = \sqrt{\frac{16}{100}} = \frac{\sqrt{16}}{\sqrt{100}} = \frac{4}{10} = \frac{2}{5} = 0.4$
**Example 1.3.90** $\sqrt{0.0016} = \sqrt{\frac{16}{10000}} = \frac{\sqrt{16}}{\sqrt{10000}} = \frac{4}{100} = 0.04$

In Excel square root function is called =SQRT(). Example 1.5.90 in Excel is as follows:

| | A | B | C |
|---|---|---|---|
| 1 | 0.0016 | 0.04 | =SQRT(A1) |

## Indices

The index is the power of the base. If we show $2^3$, then 2 is the base and 3 is the index. $2^3$ is said 'two to the power three' or 'two cubed' and means $2 \times 2 \times 2 = 8$. Equally, $3^2$ is said 'three to the power two' or 'three squared' and means $3 \times 3 = 9$ '. Now look at these examples with indices (plural of index!)

**Example 1.3.91** $10^4$ is said 'ten to the power 4' means $10 \times 10 \times 10 \times 10 = 10000$
**Example 1.3.92** $5^3$ is said '5 cubed' means $5 \times 5 \times 5 = 125$
**Example 1.3.93** $a^4$ is said 'a to the power 4' means $a \times a \times a \times a$
**Example 1.3.94** $g^6$ is said 'g to the power 6' means $g \times g \times g \times g \times g \times g$
**Example 1.3.95** $Z^7$ is said 'z to the power 7' means $z \times z \times z \times z \times z \times z \times z$.
Remember $X^1 = X$ and $X^0 = 1$.

In excel, powers are calculated as either =x^y or using the function =POWER(x,y). Example 1.5.92 is calculated as:

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 2 | 5 | 3 | 125 | =A2^B2 | |
| 3 | | | 125 | =POWER(A2,B2) | |

The following examples illustrate the rules which apply to indices.

## Multiplying numbers or letters with powers

**Example 1.3.96**

$2^3 \times 2^2 = 2 \times 2 \times 2 \times 2 \times 2 = 32$

Note: This can be written as $2^3 \times 2^2 = 2^5$

A quick way of doing this is to add the powers when you are multiplying numbers with powers.

**Example 1.3.97**

$a^6 \times a^3 = a^{6+3} = a^9$

**Example 1.3.98**

$b^6 \times b = b^{6+1} = b^7$.

Note: $b^1$ is the same as b.

**Example 1.3.99** $3k^2 \times 4k$

Written out fully, this is $3k^2 \times 4k = 3 \times k \times k \times 4 \times k = 12 \times k \times k \times k = 12k^3$

## Division of numbers or letters with powers

**Example 1.3.100**

$$\frac{2^3}{2^2} = \frac{2^1 \times 2^1 \times 2^1}{2^1 \times 2^1}$$

Now cancel, which gives 2' which is the same as 2. A quick way to divide letters or numbers with powers is to subtract the powers.

$$\frac{2^3}{2^2} = \frac{2^1 \times 2^1 \times 2^1}{2^1 \times 2^1} = 2^{3-2} = 2^1 = 2$$

**Example 1.3.101**

$$\frac{a^6}{a^3} = \frac{a^1 \times a^1 \times a^1 \times a^1 \times a^1 \times a^1}{a^1 \times a^1 \times a^1} = a^{6-3} = a^3$$

**Example 1.3.102**

$$\frac{b^4}{b} = \frac{b^1 \times b^1 \times b^1 \times b^1}{b^1} = b^{4-1} = b^3$$

Remember $b^1$, is the same as b.

**Example 1.3.103**

$$\frac{12s^2}{4s} = \frac{12 \times s^1 \times s^1}{4 \times s^1} = \frac{12}{4} s^{2-1} = 3s$$

## Brackets

**Example 1.3.104** $(a^2)^3 = a^2 \times a^2 \times a^2 = a^{2+2+2} = a^6$. A quick way of doing this is to multiply the power inside bracket by the power outside the bracket $a^{2*3} = a^6$. You get the same answer, but the latter method is quicker!

**Example 1.3.105** $(b^4)^5 = b^{4*5} = b^{20}$
**Example 1.3.106** $(2b^3)^2 = (2b^3) * (2b^3) = 4b^{3+3} = 4b^6$
**Example 1.3.107** $(4^2b^1)^2 = (4^2b^1) \times (4^2b^1) = 4^2 \times 4^2 \times b^1 \times b^1 = 256\ b^2$
**Example 1.3.108** $2(a^3)^4 = 2\ a^{3\times4} = 2\ a^{12}$

## Negative indices

**Example 1.3.109**

$9^{-1}$ means $\frac{1}{9^1}$ or $\frac{1}{9}$.

$9^{-1}$ can be written as $\frac{1}{9}$ and is known as the reciprocal of nine. When you see a negative power think 'one over'.

**Example 1.3.110**

$$10^{-1} = \frac{1}{10^1} = \frac{1}{10}$$

**Example 1.3.111**

$$a^{-1} = \frac{1}{a^1} = \frac{1}{a}$$

**Example 1.3.112**

$$a^{-4} = \frac{1}{a^4}$$

**Example 1.3.113**

$$4d^{-1} = \frac{4}{1}\frac{1}{d} = \frac{4}{d}$$

**Example 1.3.114**

$$4d^{-3} = \frac{4}{1}\frac{1}{d^3} = \frac{4}{d^3}$$

In these examples the minus sign only applies to the letter d not to the number 4.

## Fractional indices

**Example 1.3.115**

$16^{\frac{1}{2}} = \sqrt{16^1 = 4}$. This means the 'square root' of 16, is, which number multiplied by itself gives 16 - answer is 4. When you see fractional indices think 'root sign'.

**Example 1.3.116**

$8^{\frac{1}{3}} = \sqrt[3]{3^1} = \sqrt[3]{8} = 2$. This means the 'cube root' of 8, is, which number multiplied by itself three times gives 8.

To calculate the $n^{th}$ root of a number in Excel, use function =num^1/root. Example 1.3.116 is executed as:

| | A | B | C | D |
|---|---|---|---|---|
| 4 | 8 | 3 | 2 | =A4^(1/B4) |

**Example 1.3.117**

$32^{\frac{1}{5}} = \sqrt[5]{32^1} = \sqrt[5]{32} = 2$. This means the 'fifth root' of 32, is, which number multiplied by itself five times gives 32.

**Example 1.3.118**

$81^{\frac{1}{4}} = \sqrt[4]{81^1} = \sqrt[4]{81} = 3$. This means the 'fourth root' of 81, is, which number when multiplied by it self four times gives 81.

**Example 1.3.119**

$$a^{\frac{1}{3}} = \sqrt[3]{a^1} = \sqrt[3]{a}$$

**Example 1.3.120**

$$d^{\frac{1}{2}} = \sqrt[2]{d^1} = \sqrt{d}$$

**Example 1.3.121**

$27^{\frac{2}{3}}$. This should be done without a calculator. Firstly, work out the cube root of 27 = $\sqrt[3]{27} = 3$. Then square this number ($3^2$) to give a final answer of 9.

**Example 1.3.122**

$$16^{\frac{3}{4}} = \left(\sqrt[4]{16}\right)^3 = (2)^3 = 8$$

**Example 1.3.123**

$$100^{\frac{2}{3}} = \left(\sqrt[2]{100}\right)^3 = (10)^3 = 1000$$

## Fractional and negative indices

**Example 1.3.124**

$$16^{-\frac{1}{2}} = \frac{1}{16^{\frac{1}{2}}} = \frac{1}{\sqrt{16}} = \frac{1}{4}$$

**Example 1.3.125**

$$16^{-\frac{3}{4}} = \frac{1}{16^{\frac{3}{4}}} = \frac{1}{\sqrt[4]{16^3}} = \frac{1}{2^3} = \frac{1}{8}$$

**Example 1.3.126**

$$100^{-\frac{3}{2}} = \frac{1}{100^{\frac{3}{2}}} = \frac{1}{\sqrt[2]{100^3}} = \frac{1}{10^3} = \frac{1}{1000}$$

**Example 1.3.127**

$$8^{\frac{1}{2}} \times 8^{\frac{3}{2}} = 8^{\frac{1}{2}+\frac{3}{2}} = 8^{\frac{4}{2}} = 8^2 = 64$$

**Example 1.3.128**

$$8^{-\frac{1}{2}} \times 8^{\frac{1}{2}} = 8^{-\frac{1}{2}+\frac{1}{2}} = 8^0 = 1$$

**Example 1.3.129**

$$\frac{3^3 \times 3^2}{3^8} = \frac{3^5}{3^8} = 3^{-3} = \frac{1}{3^3} = \frac{1}{27}$$

***Student Exercise X1.3.23:***

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1. | a4 × a3 | 2. | b5 × b | 3. | 3s × 4s5 | 4. | 33 |
| 5. | 3-3 | 6. | 82 | 7. | 8-2 | 8. | 25 |
| 9. | 2-5 | 10. | $16^{\frac{1}{2}}$ | 11. | $9^{\frac{1}{2}}$ | 12. | $32^{\frac{1}{5}}$ |
| 13. | $27^{\frac{1}{3}}$ | 14. | $81^{\frac{1}{4}}$ | 15. | $81^{-\frac{1}{4}}$ | 16. | $8^{\frac{2}{3}}$ |
| 17. | $8^{-\frac{2}{3}}$ | 18. | $100^{-\frac{3}{2}}$ | 19. | $125^{\frac{2}{3}}$ | 20. | $a^0$ |
| 21. | $8y^0$ | 22. | $4^{\frac{1}{2}} \times 4^{\frac{3}{2}}$ | 23. | $9^{-\frac{1}{2}} \times 9^{\frac{1}{2}}$ | 24. | $9^{-\frac{3}{2}} \times 9^{\frac{1}{2}}$ |

## Standard form

Very large and very small numbers must sometimes be expressed in **standard form**, $A \times 10^n$, where 1 < A < 10 and n is an integer. Translated this means that A must be a number between 1 and 10 and n is a positive or negative number. Here are some examples to clarify this.

**Example 1.3.130** The number 87000 can be written in standard form as $8.7 \times 10^4$.

**Example 1.3.131** The number 0.000026 can be written in standard form as $2.6 \times 10^{-5}$. If we talk about the decimal point moving to give us a number between 1 and 10, you will see that if the point moves to the left, the power is positive. If the point moves to the right, the power is negative. The number of places the point moves gives us the number in the power.

**Example 1.3.132**

$$146.2 = 1.462 \times 100 = 1.462 \times 10^2$$

**Example 1.3.133**

$$26 = 2.6 \times 10 = 2.6\ 10^1$$

**Example 1.3.134**

$$0.9 = 9/10 = 9.0 \text{ X } 10^{-1}$$

**Example 1.3.135**

$$265 = 2.65 \times 100 = 2.65 \text{ x } 10^2$$

**Example 1.3.136**

$$0.0095 = 9.5 \times 10^{-3}$$

**Example 1.3.137**

$5 \times 10^{-5} \times 3 \times 10^2$. First multiply the numbers without powers ($5 \times 3 = 15$) and secondly, multiply the number with powers ($10^{-5} \times 10^2 = 10^{-5+2} = 10^{-3}$).

$$\therefore \quad 5 \times 10^{-5} \times 3 \times 10^2 = 15 \times 10^{-3} = 1.5 \times 10^{-2}$$

***Student Exercise X1.3.24:***

Write these in standard form:

| | | | | | |
|---|---|---|---|---|---|
| 1. | 6500 | 2. | 0.0082 | 3. | 132.3 |
| 4. | 0.5 | 5. | 43 | 6. | 2660000 |
| 7. | 0.35 | 8. | $0.7\times10^5\times3\times10^4$ | 9. | $6\times10^3\times2\times10^{-2}$ |
| 10. | $9\times10^{-1}\times3\times10^{-1}$ | | | | |

## Very simple equations

**Example 1.3.138** Here are some examples of very simple equations:

- $2a = 8$
- $a + 2 = 4$
- $3a + 6 = a + 10$
- $3(a + 2) = 9$
- $4a - 11 = 5a + 19$

- 2(a - 1) - 4(3a + 6) = 10
- $\frac{b}{2} = 3$
- $\frac{3a}{4} = 6$

In all these examples, there are letters and numbers on both sides of the equals sign and the letters have no powers higher than 1. (i.e. there are no $a^2$ or $a^3$ or $b^2$ or $b^3$ terms). Your answer must have a letter, which must be positive, on one side of the = sign, and a number on the other side. It does not matter which side of the equals sign the letter is! To solve simple equations, you must follow a set of rules.

**Example 1.3.139** Solve 3(a + 2) = 9

Step 1 Remove any brackets by multiplying them out.

3(a + 2) = 9

3a + 6 = 9

Step 2 Put all the terms containing letters on one side and numbers on the other side.

3a = 9 - 6

When a term 'goes over' the = sign to the opposite side, the sign changed:

| | |
|---|---|
| + becomes - | - becomes + |
| × becomes ÷ | ÷ becomes × |

3a = 9 - 6

3a = 3

3 × a = 3

a = $\frac{3}{3}$ = 1

Check by substituting a=1 into equation 3(a + 2).

3a + 6 = 9

If a = 1

3(1) + 6 = 3 + 6 = 9

Working through the examples given at the beginning of this section:

**Example 1.3.140** Solve 2a = 8

2a = 8

$a = \frac{8}{2} = 4$

Check

2a = 8

If a = 4 then 2a = 2 × 4 = 8 Correct

**Example 1.3.141** Solve a + 2 = 4

a + 2 = 4

a = 4 – 2 = 2

Check

If a = 2 then a + 2 = 2 + 2 = 4 Correct

**Example 1.3.142** Solve 3a + 6 = a + 10

3a + 6 = a + 10

3a - a = 10 - 6

2a = 4

$a = \frac{4}{2} = 2$

**Example 1.3.143** Solve 4a - 11 = 5a + 19

4a - 11 = 5a + 19

4a - 5a = +19 + 11

- 1a = 30

$a = \frac{30}{-1}$

a = - 30

**Example 1.3.144** Solve 2(a - 1) - 4(3a + 6) = 10

2(a - 1) - 4(3a + 6) = 10

2a - 2 - 12a - 24 = 10

2a - 12a = 10 + 2 + 24

- 10a = 36

$a = \frac{36}{-10} = -3.6$

**Example 1.3.145** Solve $\frac{b}{2} = 3$

$\frac{b}{2} = 3$

b = 3×2 = 6

**Example 1.3.146** Solve $\frac{3a}{4} = 6$

$\frac{3a}{4} = 6$

$a = \frac{6 \times 4}{3} = 8$

***Student Exercise X1.3.25:***

Solve:

| | | | | | |
|---|---|---|---|---|---|
| 1. | 3a = 12 | 2. | x + 3 = 7 | 3. | b - 2 = 5 |
| 4 | $\frac{b}{3} = 4$ | 5. | 2a + 5 = 9 | 6. | 5a - 3 = 22 |
| 7. | 3a + 11 = 35 – a | 8. | 4(g + 1 ) = 8 | | |
| 9. | 3(b - 1 ) - 2(3b - 2) = 4 | 10. | 4(a - 5) = 7 - 5(3 - 2a) | | |

## Substituting numbers in formulae

A formula is an equation which gives a relationship between two or more quantities e.g. c = hd, gives a formula for c in terms of h and d. c is the subject of the formula. The value of c may be found by simple arithmetic after substituting the given values of h and d. Read through the following examples.

**Example 1.3.147** If R = CA, find R when C = 6 and A =2

R = C x A

Substituting the numbers for the letters we get:

R = 6 x 2 = 12

**Example 1.3.148** If v = u + at, find v when u = 10, a=2, t=6

v = u + (a × t)

Substituting the numbers for the letters we get:

v = 10 + (2 × 6)

v = 10 + 12 = 22

**Example 1.3.149** If $I = \frac{PRT}{100}$, find I, when P = 500, R = 3, T = 2

$I = \frac{PRT}{100}$

Substituting the numbers for letters we get:

$I = \frac{500 \times 3 \times 2}{100} = 30$

**Example 1.3.150** If $W = \frac{kz^2}{3}$, find W, when k = 9 and z = 5

Substituting the numbers for letters we get

$W = \frac{kz^2}{3}$

$W = \frac{9 \times 5^2}{3}$

$W = \frac{9 \times 25}{3}$=75

**Example 1.3.151** If C = 30(R - 2), find C when R = 6

Substituting the numbers for letters we get:

C = 30(6 - 2)

C = 30×4 = 120

Remember to work out the bracket first!

**Example 1.3.152** Find R from the formula P = RT when P = 20 and T = 4

P = RT

Substituting the numbers for letters we get:

20 = R × 4

R = $\frac{2}{4}$ = 5

**Example 1.3.153** Find 'a' from the formula S = Ta + b, when S = 60, b = 12 and T = 8.

S = Ta + b

Substituting numbers for letters:

60 = 8a+ 12

60 - 12 = 8a

48 = 8a

$a = \frac{48}{8} = 6$

**Example 1.3.154** Find x from the formula A = xyz, when A = 80, y = 4 and z = 5.

A = xyz

Substituting numbers for letters we get:

$80 = x \times 4 \times 5$

80 = 20x

$x = \frac{80}{20} = 4$

**Example 1.3.155** If C = 2(R - 6) find R when C = 24.

C = 2(R - 6)

Substituting the numbers for letters:

24 = 2(R-6)

24 = 2R - 12

24+12 = 2R

36 = 2R

$\frac{36}{2} = R$

R = 18

***<u>Student Exercise X</u>*<u>1.3.</u>*<u>26:</u>***

1. If J = ak, find J, when a = 15, and k = 3.
2. If P = r - st, find P. when r = 20, s = 2 and t = 3.
3. If $I = \frac{PRT}{100}$, find I, when P = 200, and R = 4 end T = 2.
4. If $x = \frac{pz^2}{2}$, find x when p = 1 and z = 6.
5. If C = 20(z + 6), find C, when z = 2.
6. Find R from the formula, Z = RY, when Z = 40, and Y =5.
7. Find A from the formula J = BA + C, when J = 120, C = 12 and B = 8.
8. Find C from the formula H = Cbn, when H = 100, b = 2, and n = 10.
9. If R = 3(p - 2), find R. when p = 9.
10. If $C = \frac{2j^2}{k}$, find C when j = 3, and k = 6.

# Exercise Answers

1. Number answers to be included here.
2. Detailed solutions accessible online via the instructor manual.

Exercise XO.1

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1. | 80% | 2. | $33\frac{1}{3}\%$ | 3. | 75% | 4. | 28 4/7 % |
| 5. | 120% | | | | | | |

Exercise XO.2

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1. | 65% | 2. | 37. 5% | 3. | 89% | 4. | 60% |
| 5. | 234% | | | | | | |

Exercise XO.3

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1. | ¼ | 2. | 3/10 | 3. | 1 2/5 | 4. | 1/3 |
| 5. | 3/8 | | | | | | |

Exercise XO.4

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1. | 0.63 | 2. | 0.047 | 3. | 0.5165 | 4. | 0.031 |
| 5. | 0.07 | | | | | | |

Exercise XO.5

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1. | 15 | 2. | 21 | 3. | 15 | 4. | 3.6 |
| 5. | 30 | | | | | | |

Exercise XO.6

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1. | £208 | 2. | £192 | 3. | £462 | 4. | £378 |

Exercise XO.7

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1. | 25% | 2. | 20% | 3. | 28 4/7% | 4. | 20% |
| 5. | 12 1/2% | | | | | | |

Exercise XO.8

1. (a) 1 : 2 (b) 2 : 3 (c) 2 : 5 (d) 6 : 5 (e) 8 : 1
2. 20:3
3. 1 : 5 as a fraction $\frac{1}{5}$

Exercise XO.9

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1. | £200 : £100 | | | 2. | 25 : 35 | 3. | 14cm, 21 cm, 49 cm |
| 4. | (i) £12 | (ii) | £30 | 5. | £210 | | |

Exercise XO.10

| | | | | | |
|---|---|---|---|---|---|
| 1. | 60p | 2. | 64p | 3. | 7.5 hours or 7 hours 30 minutes |
| 4. | £5.25 | 5. | 20 glasses | | |

Exercise XO.11

1. €248.56 2. £482.78 3. £187.50 4. 400 000 lire

5. (i) 30000 (ii) £100.00

Exercise XO.12

| | | | | | |
|---|---|---|---|---|---|
| 1. | 6 days | 2. | £6 | 3. | 4 days |

Exercise XO.13

| | | | |
|---|---|---|---|
| 1. | + 3 + 9 = + 12 | 2. | + 10 - 5 = + 5 |
| 3. | - 15 + 2 = - 13 | 4. | - 20 - 20 = - 40 |
| 5. | + 13 - 10 = + 3 | 6. | + 24 + 12 = + 36 |
| 7. | - 21 - 21 = - 42 | 8. | - 21 + 21 = 0 |
| 9. | + 12 - 12+ 12 = + 12 | 10. | + 100 + 50 + 20 =·+ 170 |

Exercise XO.14

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1. | + 12 | 2. | + 12 | 3. | + 40 | 4. | + 9 |
| 5. | + 16 | 6. | - 36 | 7. | + 3 | 8. | + 3 |
| 9. | - 3 | 10. | - 100 | 11. | - 2 | 1O. | - 6 |
| 13. | - 4 | 14. | - 51/3 | 15. | - 4 2/3 | 16 | + 25 |

Exercise XO.15

Here a, b and c have been chosen for the numbers.

| | | | | | |
|---|---|---|---|---|---|
| 1. | 4a | 2. | $\frac{a}{4}$ | 3. | a-8 |
| 4 | a + 6 | 5. | a + b + c | 6. | 3ab |
| 7. | 6a + 5b | 8. | 4a - b | | |

Exercise XO.16

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1. | 5 | 2. | 2 | 3. | 7 | 4. | 2 |
| 5. | 18 | 6. | 0 | 7. | 6 | 8. | 24 |
| 9. | 24 | 10. | 10 | | | | |

Exercise XO.17

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1. | 9 | 2. | 64 | 3. | 50 | 4. | 12 |
| 5. | 76 | 6. | 58 | 7. | 30 | 8. | 100 |
| 9. | 4.5 | 10. | 12.5 | | | | |

Exercise XO.18

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1. | 4 | 2. | - 6 | 3. | 2 | 4. | 10 | 5. | 0 |

Exercise XO.19

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1. | 14a | 2. | 5a | 3. | 8xy | 4. | 7a |
| 5. | 6b | 6. | 2a + 2b | 7. | 6b - 13a | 8. | $8b^2 + 2b$ |
| 9. | $9s^2t$ | 10. | $8b^2 + 7b$ | | | | |

Exercise XO.20

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1. | $6a^2$ | 2. | 6ab | 3. | 24a | 4. | $- 12a^2$ |
| 5. | - 12st | 6. | +12st | 7. | $8a^4$ | 8. | $6a^2b^2$ |
| 9. | ab | 10. | 4a | 11. | $\frac{2a}{b}$ | 12. | 4abc |
| 13. | $\frac{-a}{b}$ | 14. | $\frac{3}{y}$ | 15. | $\frac{1}{3}$ | | |

Exercise XO.21

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1. | 2x + 6 | 2. | 4a + 4b | 3. | 6a - 6b | 4. | 5x - 15 |
| 5. | 12x + 6y | 6. | - m - n | 7. | - 6x - 10 | 8. | - 12 + 18x |
| 9. | - 2p - 3q | 10. | $12ab - 4b^2$ | | | | |

Exercise XO.22

| | | | | | |
|---|---|---|---|---|---|
| 1. | 5x + 16 | 2. | x - 10 | 3. | $- 2x^2 + x - 4$ |
| 4. | 8a | 5. | $7x^2 + 29x + 16$ | | |

Exercise XO.23

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1. | $a^7$ | 2. | $b^6$ | 3. | $12s^6$ | 4. | 27 |
| 5. | $\frac{1}{27}$ | 6. | 64 | 7. | $\frac{1}{64}$ | 8. | 32 |
| 9. | $\frac{1}{32}$ | 10. | 4 | 11. | 3 | 12. | 2 |
| 13. | 3 | 14. | 3 | 15. | $\frac{1}{3}$ | 16. | 4 |
| 17. | $\frac{1}{4}$ | 18. | $\frac{1}{1000}$ | 19. | 25 | 20. | 1 |
| 21. | 8 | 22. | 16 | 23. | 1 | 24. | $\frac{1}{9}$ |

Exercise XO.24

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1. | $6.5 * 10^3$ | 2. | $8.2 * 10^{-3}$ | 3. | $1.323 * 10^2$ | 4. | $5 * 10^{-1}$ |
| 5. | 4.3 * 106 | 6. | $2.66 * 10^6$ | 7. | $3.5 * 10^{-1}$ | 8. | $2.1 * 10^9$ |
| 9. | $1.2 * 10^2$ | 10. | $2.7 * 10^{-1}$ | | | | |

Exercise XO.25

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1. | a = 4 | 2. | x = 4 | 3. | b = 7 | 4. | b = 12 |
| 5. | a = 2 | 6. | a = 5 | 7. | a = 6 | 8. | g = 1 |
| 9. | b = -1 | 10. | a = -2 | | | | |

Exercise XO.26

1. J = 45 2. P = 14 3. I = 16 4. x = 18
5. C = 160 6. R = 8 7. A = 13.5 8. C = 5
9. R = 21 10. C = 3

# Part 2 Advanced maths skills (with emphasis on Excel)

## 2.1 Summation

The Greek capital letter sigma Σ is used as a symbol for summation. If the values of variable x are: 1, 4, 8 and 7, then Σx = 1 + 4 + 8 + 7 = 20. In Excel, summation is executed using the function =SUM(range).

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | **Summation** | | | | | |
| 2 | 1 | | | 1 | =A2^2 | |
| 3 | 4 | | | 16 | =A3^2 | |
| 4 | 8 | | | 64 | =A4^2 | |
| 5 | 7 | | | 49 | =A5^2 | |
| 6 | 20 | =SUM(A2:A5) | | 130 | =SUM(D2:D5) | |
| 7 | | | | 130 | =SUMSQ(A2:A5) | |

Instead of saying that x variable is 1, 4, 8 and 7, we could have used the notation: $x_1$=1, $x_2$=4, $x_3$=8 and $x_4$=7. This subscript notation enables us to write an expression, such as: $\sum_{i=1}^{n} x_i$. Our example becomes: $\sum_{i=1}^{4} x_i = x_1 + x_2 + x_3 + x_4 = 1 + 4 + 8 + 7 = 20$.

Assuming that we have the same values of x: 1, 4, 8 and 7, then the same logic, for example, applies to the expression: Σ(x-3) = (1-3) + (4-3) + (8-3) + (7-3) = -2 + 1 + 5 + 4 = 8. However, note that this is not the same as Σx-3 = 20 - 3 = 17.

The same summation principle also applies to: $\Sigma x^2 = 1^2 + 4^2 + 8^2 + 7^2$ = 1 + 16 + 64 + 49 = 130. In Excel we can either first calculate the squared values of every number and then add them up using =SUM() function, or we can use the =SUMSQ() directly (see cell D7 above).

If we have two data sets x and y, then to calculate Σxy, given that x 1, 4, 8 and 7, and y is 2, 4, 7 and 3: Σxy = 1×2 + 4×4 + 8×7 + 7×3 = 2 + 16 + 56 + 21 = 95.

The above operation is executed in Excel as follows:

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 8 | **Summation 2 variables** | | | | | |
| 9 | **x** | **y** | | | | |
| 10 | 1 | 2 | 2 | =A10*B10 | | |
| 11 | 4 | 4 | 16 | =A11*B11 | | |
| 12 | 8 | 7 | 56 | =A12*B12 | | |
| 13 | 7 | 3 | 21 | =A13*B13 | | |
| 14 | | | 95 | =SUM(C10:C13) | | |
| 15 | | | 95 | =SUMPRODUCT(A10:A13,B10:B13) | | |

The alternative is to use Excel function =SUMPRODUCT() as in cell C15 above).

If the mean value of our mini-series (1, 4, 8, 7) is $\bar{x} = \frac{\sum x}{n} = \frac{1+4+8+7}{4} = \frac{20}{4} = 5$, then if we have $\sum(x - \bar{x})$, the result is = (1-5) + (4-5) + (8-5) + (7-5) = -4 - 1 + 3 + 2 =0. In fact, this particular expression is equal to zero for any data set.

## 2.2 Product

The Greek capital letter pi Π is used as a symbol for a product. If, again, the values of variable x are: 1, 4, 8 and 7, then Πx means: Πx = 1 × 4 × 8 × 7 = 224. Just as with the example of summation, we can express the product as $\prod_{i=1}^{n} x_i$, which is calculated as $\prod_{i=1}^{4} x_i = 1 \times 4 \times 8 \times 7 = 224$.

Excel function for product is =PRODUCT(range). Below is the example of how to use it.

| | A | B | C |
|---|---|---|---|
| 17 | **Product** | | |
| 18 | 1 | | |
| 19 | 4 | | |
| 20 | 8 | | |
| 21 | 7 | | |
| 22 | 224 | =PRODUCT(A18:A21) | |

## 2.3 Factorial

The exclamation sign is a symbol for the factorial function. For example: 4! = 4 × 3 × 2 × 1 = 24. Or in general n!=(n-1) × (n-2) × … 1. Remember that 1!=1 and that 0!=1. Other than that, factorials are very intuitive.

One way to express the factorial is to use the expression: $n! = \prod_{k=1}^{n} k$.

Excel function for factorial is =FACT(number). Below is the example how to use it:

| | A | B | C |
|---|---|---|---|
| 24 | **Factorial** | | |
| 25 | 5 | 120 | =FACT(A25) |

## 2.4 Binomial coefficient

If we see two numbers one above the other, but without a fraction bar, and they are inside the brackets, like this: $\binom{n}{r}$, this is called a binomial coefficient. This is calculated as the number of combinations of *n*, taken *r* at a time. For example, $\binom{5}{2}$=10, because the formula is: $\binom{n}{r} = \frac{n!}{r!(n-r)!}$, which in our case is: $\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5!}{2!\,3!} = \frac{5\times4\times3\times2\times1}{2\times1 \times 3\times2\times1} = \frac{120}{2 \times 6} = 10$.

Imagine you had pieces of paper with one letter between a to e printed on each piece of paper. How many combinations of two letter you can select, providing that you do not allow the repeats (such as that a,b is the same as b,a and therefore not permitted)? The answer is 10, as we calculated above, and below we show all the combinations.

| | | | |
|---|---|---|---|
| a,b,c,d,e | | | |
| a,b | b,c | c,d | d,e |
| a,c | b,d | c,e | |
| a,d | b,e | | |
| a,e | | | |

Excel function used to calculate the binomial coefficient is =COMBIN(n,r). Below is an example:

| | A | B | C | D |
|---|---|---|---|---|
| 27 | **Binomial coefficient** | | | |
| 28 | 5 | | | |
| 29 | 2 | | | |
| 30 | | 10 | =COMBIN(A28,A29) | |
| 31 | | | | |

## 2.5 Pi (π) and *e*

Pi, written as the Greek latter π, is an irrational number and in its shortest form it is known as 3.14. However, its decimal form never ends (for example, a number that looks similar to π, such as 3.1415900000 clearly ends after the fifth decimal point), nor it ever becomes repetitive (like 3.141599999). It has an infinite number of digits behind the decimal point and there is no pattern among them.

π represents the ratio of the circumference of any circle to the diameter of that circle. Regardless of the circle's size, this ratio will always be equal to π.

Below we are using several examples in Excel. The diameters vary from 7, 14, 28 and 12. Their corresponding circumferences are 22, 44, 88 and 37.7. If we divide the circumference by the diameter, we always get 3.14, which is the value of π.

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 33 | **PI** | | | | |
| 34 | Circumference | | Diameter | π | |
| 35 | 22 | | 7 | 3.14 | =A35/C35 |
| 36 | 44 | | 14 | 3.14 | =A36/C36 |
| 37 | 88 | | 28 | 3.14 | =A37/C37 |
| 38 | 37.7 | | 12 | 3.14 | =A38/C38 |

Excel also has a function dedicated to π and it is =PI(). In cell A8, we just typed =PI() and Excel shows the value of 3.141593.

| | A | B |
|---|---|---|
| 40 | 3.141593 | =PI() |

The maximum number of digits that Excel will display for =PI() function is 14. If you extend the field in which $\pi$ is displayed, after the fourteenth digit behind the decimal points you will see just zeros: 3.14159265358979000000.

Another famous irrational number is *e*. It is often also called Euler's number in tribute to the famous mathematician Euler. The value of *e* is approximately 2.7182… . Just like $\pi$, it has an infinite number of digits behind the decimal point and there is no pattern among them. Also, *e* is the base of the natural logarithms (see below section 2.7).

The value of *e* is in Excel obtained via the function =EXP(number). If we put 1 as the number, the value is 2.7182… . Below is the example from Excel:

| | A | B |
|---|---|---|
| 42 | *e* | |
| 43 | 2.718282 | =EXP(1) |

## 2.6 Compound interest and *e*

There is a connection between the compound interest calculations in finance and the number *e*. In its briefest form, the formula for the compound interest is $(1+r/n)^n$, where r is the annual interest rate and n is the number of periods within the year. If interest is paid at the following intervals, we get some interesting results:

| | I | J | K | L |
|---|---|---|---|---|
| 10 | **Interest** | **Formula** | | |
| 11 | 2 | 2.25 | =(1+(1/I11))^I11 | |
| 12 | 12 | 2.61303529 | =(1+(1/I12))^I12 | |
| 13 | 52 | 2.692596954 | =(1+(1/I13))^I13 | |
| 14 | 365 | 2.714567482 | =(1+(1/I14))^I14 | |
| 15 | 8760 | 2.718126692 | =(1+(1/I15))^I15 | |
| 16 | 100000 | 2.718268237 | =(1+(1/I16))^I16 | |
| 17 | etc. | 2.718281828 | =EXP(1) | |

In the series of examples, we are paying interest at first every half year, i.e. twice a year (row 11), then once a month, then once a week, then once a day, then once every hour (row 15), etc. We also show the results if we paid every 100,000$^{th}$ interval (row 16). We can see that the results converge towards 2.718, which is *e*. This is just one of many examples where *e* as an irrational number appears. We could find at least a dozen of other applications from engineering to physics.

## 2.7 Powers, exponents and logarithms

To square a number, we just multiply it by itself, for example 2.828427×2.828427=7.9999…≈8. This is effectively using the number to the power of 2, which is called the exponent. In Excel we would use a symbol: 2.828427^2=8. The opposite operation is taking the square root, so to go back from 8, we use Excel function =SQRT(8), which will give us 2.828427.

However if the exponent is larger than 2, then we cannot use the =SQRT() function. Excel does not have a dedicated function for calculating roots beyond number 2. The alternative is to use =8^(1/2), which is $8^{\frac{1}{2}}$. This will also give us 2.828427. The third option is to use Excel function =POWER(8,0.5), where 0.5 is just another expression for $\frac{1}{2}$.

If we use another example, shown below, where $8^3$=512, which in Excel is expressed as =POWER(8,3). What happens if we wanted to calculate the third root of the value of 8? This is $\sqrt[3]{8}$ or $8^{\frac{1}{3}}$. In Excel syntax, this is =8^(1/3). The value we get is 2. This value of 2 has a special meaning.

| | I | J | K | L | M |
|---|---|---|---|---|---|
| 19 | **Powers and exponents** | | | | |
| 20 | 2.828427 | 2 | 8 | =I20^J20 | |
| 21 | | | 2.828427 | =SQRT(I22) | |
| 22 | 8 | 2 | 2.828427 | =I22^(1/J22) | |
| 23 | 8 | 0.5 | 2.828427 | =POWER(I23,J23) | |
| 24 | | | | | |
| 25 | 8 | 3 | 512 | =I25^J25 | |
| 26 | | | 512 | =POWER(I25,J25) | |
| 27 | 8 | 3 | 2 | =I27^(1/J27) | |

Effectively $2^3$ is 8 (2×2×2), which means that we multiplied 2 by itself 3 times to get this result. This number 3, in this case, is called the logarithm. This can be written as $\log_2(8)=3$. In this case, 2 is the base, 8 is the argument and the logarithm is 3. If we had $\log_5(625)=4$, then the base is 5, argument is 625 and the logarithm is 4 (because 5×5×5×5=625). In general, if $b^x=n$, then $\log_b(n)=x$. As we can see, taking logarithms is inverse to calculating the exponent of the base.

If the logarithm is written without the base, such as log(100)=2, then it is assumed that the base is 10 (because $10^2$=100). These are called the common logarithms.

If the logarithm is written as $\log_e(7.389)\approx2$, then this is called a natural logarithm ($e^2=2.718^2=7.389$). In Excel, several functions are used to retrieve logarithms. Below are some examples of using Excel functions for logarithms.

| | I | J | K | L | M |
|---|---|---|---|---|---|
| 29 | **Logarithms** | | | | |
| 30 | 25 | 1.397940009 | =LOG(I30) | | |
| 31 | | 1.397940009 | =LOG10(I30) | | |
| 32 | | 1.397940009 | =LOG(I30,10) | | |
| 33 | 25 | 3 | 2.929947 | =LOG(I33,J33) | |
| 34 | 25 | 3.218875825 | =LN(I34) | | |
| 35 | 1 | 2.718281828 | =EXP(I35) | | |
| 36 | 2.718282 | 1 | =LN(I36) | | |
| 37 | | | | | |
| 38 | | 3 | 20.08554 | =EXP(J38) | |
| 39 | 2.718282 | 3 | 20.08554 | =I39^J39 | |

Cell J30 contains function =LOG(). This is equivalent to the function =LOG10() from cell I30 and it returns the logarithm value for the base 10. The same result can be obtained using the function =LOG(number, base), which is what we did in cell J32, where the base was the same as before, i.e. 10. This last format of the log function in Excel can be used for any other base. In cell J33, we show the result for 25, but for the base 3. Cell J34 contains the function used for the natural logarithms, which is Excel function =LN().

We know that the value of *e* is approximately 2.718281828… . Cell J35 shows this value calculated using the function =EXP(). In cell J36, we reversed this operation by calculating the natural log value of 2.718281828. As expected, the result is 1. Remember, functions =EXP() is inverse of function =LN()

As =EXP() function is the same as saying $e^n$, we are showing in cells K38 and K39 identical results for using either =EXP(3) or =2.718282^3.

## 2.8 Complex numbers

We now revisit how to calculate the square root of a number. If, $\sqrt{16} = 4$ and $\sqrt{1} = 1$, what is $\sqrt{-1} =$ ? It turns out that the solution is the imaginary number *i*. So, the solution is: $\sqrt{-1} = \pm i$. This means that the square root of a negative number will always have the imaginary number *i* attached to the actual value of the square root of the absolute part. So, this means, for example, that $\sqrt{-9} = \pm 3i$.

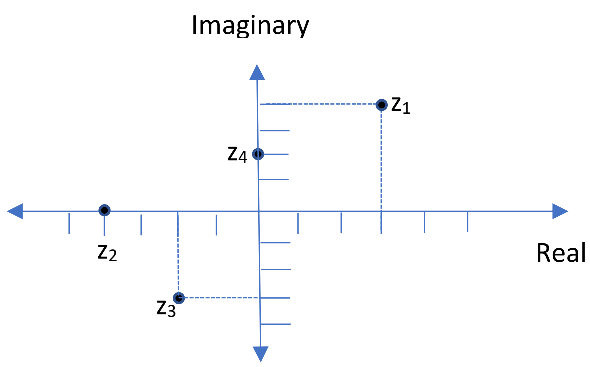
If we use Excel to calculate the square root of the number i.e. =SQRT(), using the same examples as above, we get the same results, except that the negative numbers produce #NUM! result. This means that to calculate the square root of a negative number, we must use the function =IMSQRT(number).

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | **Square roots** | | | | |
| 2 | 16 | 4 | =SQRT(A2) | | |
| 3 | 1 | 1 | =SQRT(A3) | | |
| 4 | -1 | #NUM! | =SQRT(A4) | 6.1257422745431E-17+i | =IMSQRT(A4) |
| 5 | -9 | #NUM! | =SQRT(A5) | 1.83772268236293E-16+3i | =IMSQRT(A5) |

Unfortunately Excel is not precise enough, so for =IMSQRT(-1) we get the result of: 0.0000000000000000061257422745431+*i*. As we can see in cell D4, sixteen zeros after the decimal point. The first part before *i* is virtually zero, so the result is *i*. In cell D5 in the above example we see that the square root of -9 is 3*i*. As before, the number 3*i* is preceded by 16 zeros, so for all practical purposes it is zero.

Imaginary numbers can be combined with real numbers, which then creates the complex numbers. A typical complex number is: z = x + *i*y. Specific examples are: $z_1$=3+4*i*, or $z_2$=-4, or $z_3$=-2-3*i*, or $z_4$=2*i*. Examples $z_1$ and $z_3$ consist of both the real and imaginary part, $z_2$ has only the real part and $z_4$ only the imaginary part.

If real numbers can be represented on a line, then complex numbers can be represented in a two-dimensional space, where one axis is real and the other imaginary. The above four complex number examples are depicted as:



Most of the usual algebraic laws apply to complex numbers, and they are added / subtracted / multiplied in a similar way as any regular numbers. The only difference is that you do not mix the real and imaginary part of the complex number.

Excel uses the function =COMPLEX(real,imaginary) to define a complex number. In the example below we can see that in cell C9 there is only the real part present in the complex number and in cell C10 only the imaginary part. Cell C11 shows both the real and imaginary part.

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 8 | **Complex numbers** | | | | | | |
| 9 | 1 | 0 | 1 | =COMPLEX(A9,B9) | 0 | =IMAGINARY(C9) | |
| 10 | 0 | 1 | i | =COMPLEX(A10,B10) | 1 | =IMAGINARY(C10) | |
| 11 | 1 | 1 | 1+i | =COMPLEX(A11,B11) | 1 | =IMAGINARY(C11) | |

In cells E9:E11 we get the same results, but we used the function =IMAGINARY(), which only shows the imaginary part of the complex number. To show just the real part of the complex number, we would have to use the function =IMREAL(). Example below shows both examples.

| | A | B | C | D |
|---|---|---|---|---|
| 13 | 5+2i | | 2 | =IMAGINARY(A13) |
| 14 | 5+2i | | 5 | =IMREAL(A14) |

We'll show a couple of examples how to add and multiply complex numbers.

Let's use another example. If $z_1=3+4i$ and $z_2=-2-3i$, then $z_1+z_2$ is: $z_1+z_2 = (3+(-2))+(4i-3) = 1+i$. An example from Excel uses a dedicated function for adding complex numbers =IMSUM().

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| 16 | 3 | 4 | 3+4i | =COMPLEX(A16,B16) | 3 | =IMREAL(C16) | | 4 | =IMAGINARY(C16) | |
| 17 | -2 | -3 | -2-3i | =COMPLEX(A17,B17) | -2 | =IMREAL(C17) | | -3 | =IMAGINARY(C17) | |
| 18 | | | 1+i | =IMSUM(C16,C17) | 1 | =IMREAL(C18) | | 1 | =IMAGINARY(C18) | |

Cells E16:E18 show the real part of the results from C16:C18 and cells H16:H18 show the imaginary part of the same results.

In another example, if $z_1=3+4i$ and $z_2=-4$, find $z_1+z_2$. The result is: $z_1+z_2 = (3+(-4))+(4i+0) = -1+4i$. We see the result below in Excel sheet.

| | A | B | C | D |
|---|---|---|---|---|
| 20 | 3 | 4 | 3+4i | =COMPLEX(A20,B20) |
| 21 | -4 | | -4 | =COMPLEX(A21,B21) |
| 22 | | | -1+4i | =IMSUM(C20,C21) |

Let's look at another example. If $z_3=-2-3i$ and $z_4=2i$, find $z_1 \times z_2$. The result is: $z_1 \times z_2 = (2i \times (-2)) + (2i \times (-3i)) = -4i - 6i^2$. Because $i^2=-1$, we have $-4i - 6 \times (-1) = -4i + 6$. The example below uses Excel function =IMPRODUCT() to multiply two complex numbers.

| | A | B | C | D |
|---|---|---|---|---|
| 24 | -2 | -3 | -2-3i | =COMPLEX(A24,B24) |
| 25 | 0 | 2 | 2i | =COMPLEX(A25,B25) |
| 26 | | | 6-4i | =IMPRODUCT(C24,C25) |

Just to demonstrate, if you use the same function, =IMPRODUCT(), to multiply two *i* numbers, you will get -1, which is exactly what $i^2$ is. Remember, $i^2$ = -1 and $i = \sqrt{-1}$. See below the same example in Excel.

| | A | B | C | D |
|---|---|---|---|---|
| 28 | i | i | -1 | =IMPRODUCT(A28,B28) |

Sometimes when Excel produces a result which is a complex number, it is a very long string of digits as in cell A31 below:

| | A | B | C | D | E | F | G | H | I | J | K | L |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 31 | -19.9999999999994-5.85786437626833i | | | | -20-5.858i | | | | | | | |
| 32 | | | | | =COMPLEX(VALUE(TEXT(IMREAL(A31),"0.000")),VALUE(TEXT(IMAGINARY(A31),"0.000"))) | | | | | | | |

To shorten it to a more manageable format (to three digits, for example), we can combine the function =COMPLEX() with a few text manipulation functions, such as above. The same complex number shown in cell E31 looks much more easier to read.

There is quite a long list of functions in Excel dedicated to complex numbers. They all start with the prefix IM. Here is the list:

IMABS, IMAGINARY, IMARGUMENT, IMCONJUGATE, IMCOS, IMCOSH, IMCOT, IMCSC, IMCSCH, IMDIV, IMEXP, IMLN, IMLOG10, IMLOG2, IMPOWER, IMPRODUCT, IMREAL, IMSEC, IMSECH, IMSIN, IMSINH, IMSQRT, IMSUB, IMSUM AND IMTAN.

They are all located in the engineering group of functions.

## 2.9 Relationship between π, *e* and *i*

There is a remarkable relationship that brings together $\pi$, *e* and *i*, and it is called the Euler's formula. Euler established that: $e^{i\pi}$ = -1, or $e^{i\pi}$ + 1 = 0, or $e^{2\pi\sqrt{-1}} = 1$. Amazing!

We can try to calculate the same in Excel, though the result is not perfect (we already demonstrated some imprecisions in Excel). The example below demonstrates it.

| | K | L | M | N | O | P | Q | R |
|---|---|---|---|---|---|---|---|---|
| 1 | **π, e , i** | | | | | | | |
| 2 | 2.718282 | 3.141593 | i | | -1+3.2311393144413E-15i | | | |
| 3 | =EXP(1) | =PI() | =COMPLEX(0,1) | | =IMEXP(IMPRODUCT(COMPLEX(0,1),PI())) | | | |
| 4 | | | | | | | | |
| 5 | | 6.283185 | | | 1+3.30768398781878E-15i | | | |
| 6 | | =2*L2 | | | =IMEXP(IMPRODUCT(L5,M2)) | | | |

To show in Excel that $e^{i\pi} = -1$, let's take a look at cell K2 which contains the value of *e*, cell L2 the value of $\pi$, and cell M2 just the value of *i*. The result is in shown in O2, which is the expression $e^{i\pi}$ shown in Excel syntax.

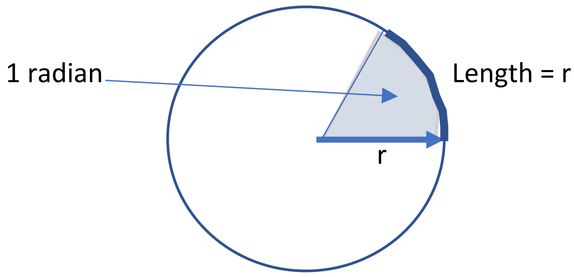
To get the result, we use =IMEXP() function, which is similar to =EXP(). Function =IMEXP() returns the value of *e* raised to the power of a complex number. As the exponent in this case is a complex number, we have to use the product function for complex numbers, which is =IMPRODUCT(). As we can see, we multiply *i* (written as =COMPLEX()) multiplied with =PI(). The result is -1+0.000000000000000032311…*i* . The first part is the real part (-1) and the second part is imaginary. The imaginary number is virtually zero (fifteen zeros before 32311…), and we can ignore it for all practical purposes. This means that effectively Excel returns the result of -1, which is to be expected.

To show in Excel that $e^{2\pi\sqrt{-1}}$, let's take a look at cell L5 that contains the value of $2\pi$. Cell O5 produces the final result and we used the same functions as in the previous paragraph. We can see the result is 1+0.00000000000000330768…*i* . As before, the imaginary part is virtually zero and the result is +1, as expected. Again, not perfect, but given Excel internal engine, good enough!

## 2.10 Trigonometry

The angles are usually measured in degrees. The right angle is $90^0$, for example, and the full circle consists of $360^0$, as we all know. However, there is another unit to measure angles, and it is called radians. When using Excel, radians are also a key unit for handling angles, so we better understand how they work and how they relate to degrees.

If we take a radius of a circle and wrap the length of this radius around the circle, then we can "connect" every end of this length on the circle with the centre of the circle. The angle that such formed triangle has created is 1 radian. The picture below shows the idea that we are trying to convey.




In a way, radian is a true measure of the circle, because it is defined by the radius of the circle.

In a half circle, there are $\pi$ number of radians. As the value of $\pi$ is 3.14…, this means that half circle is the same as saying 3.14 radians and the full circle is 6.28 radians. One way to understand this is to say that if the radius of our circle (any circle) is a piece of string of a certain length, then we would need 6.28 pieces of this string to put around the circle to completely cover the perimeter of this circle. As we know, a more precise answer is $2\pi$ pieces of string.

Clearly there is relationship between radians and degrees. We also know that half circle is $180^0$, which means that 1 radian is $180^0 / \pi$, or $180/3.14 \approx 57.295..^0$.

Now we know that one radian is approximately $57^0$, we can establish how to convert from radians into degrees, and other way round with two simple formulae.

The conversion from radians to degrees is done as:

$$Degrees = \frac{Radians \times 180}{\pi}$$

And the conversion from degrees to radians is done as:

$$Radians = \frac{Degrees \times \pi}{180}$$

We can create a little table to show how different degrees translate into radians and what is their true value when expressed in terms of $\pi$.

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | **2.10 Trigonometry** | | | | |
| 2 | Degrees | Radians (exact) | Radians (approx.) | | |
| 3 | 30 | $\pi/6$ | 0.524 | 0.524 | |
| 4 | 45 | $\pi/4$ | 0.785 | 0.785 | |
| 5 | 60 | $\pi/3$ | 1.047 | 1.047 | |
| 6 | 90 | $\pi/2$ | 1.571 | 1.571 | |
| 7 | 180 | $\pi$ | 3.142 | 3.142 | |
| 8 | 270 | $3\pi/2$ | 4.712 | 4.712 | |
| 9 | 360 | $2\pi$ | 6.283 | 6.283 | |
| 10 | | | =A9*PI()/180 | =RADIANS(A9) | |

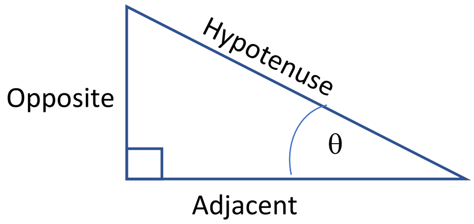
Column B shows the true value of an angle in radians expressed as a fraction of $\pi$. However, because $\pi$ is a famous irrational number that is approximated by 3.1428571…, these exact values are approximated by multiplying the degrees with $\pi$ over 180. Column C shows this formula for converting degrees into radians. However, in Excel we do not have to do this, we can use a dedicated =RADIANS() function, as in column D. The results are identical to those in column C.

In Excel the function that is opposite to =RADIANs(number) is =DEGREES(number). The table below shows a few examples of conversions from degrees to radians and from radians to degrees.

| | A | B | C |
|---|---|---|---|
| 12 | Degrees | Radians | |
| 13 | 30 | 0.523598776 | =RADIANS(A13) |
| 14 | 45 | 0.785398163 | =RADIANS(A14) |
| 15 | 90 | 1.570796327 | =RADIANS(A15) |
| 16 | 180 | 3.141592654 | =RADIANS(A16) |
| 17 | 360 | 6.283185307 | =RADIANS(A17) |
| 18 | 27.6 | 0.481710874 | =RADIANS(A18) |
| 19 | | | |
| 20 | Radians | Degrees | |
| 21 | 0.1 | 5.7 | =DEGREES(A21) |
| 22 | 0.523 | 30.0 | =DEGREES(A22) |
| 23 | 1.57 | 90.0 | =DEGREES(A23) |
| 24 | 3.14 | 179.9 | =DEGREES(A24) |
| 25 | 6.283 | 360.0 | =DEGREES(A25) |
| 26 | 0.482 | 27.6 | =DEGREES(A26) |

Now we are ready to explore some basic trigonometric functions.

We'll start with a simple right angle triangle. Take a look at the angle θ (Greek letter theta), which is opposite of the right angle, symbolised by the little square □.




The three sides are called: adjacent (a shorter side next to θ), opposite and hypotenuse (a longer one next to θ). You will remember this from elementary mathematics, because Pythagoras' theorem is usually derived in this manner. However, this is also the basis for explaining some elementary concepts in trigonometry. For any angle θ, the three basic functions of trigonometry, which are the sinus, cosines and tangent, can be calculated as:

$$\sin\theta = \frac{Opposite}{Hypotenuse}$$

$$\cos\theta = \frac{Adjecent}{Hypotenuse}$$

$$\tan\theta = \frac{Opposite}{Adjacent}$$

If, for example: Opposite=3, Adjacent=4 and Hypotenuse=5, then the three trigonometric values we listed here are calculated as:

sin θ = 3/5 = 0.6
cos θ = 4/5 = 0.8
tan θ = 3/4 = 0.75

In addition to these three basic trigonometric functions, there are three other "convenience" functions. They are nothing but the reciprocals, and they are:

$\csc\theta = \frac{1}{\sin\theta}$, $\sec\theta = \frac{1}{\cos\theta}$ and $\cot\theta = \frac{1}{\tan\theta}$

These functions are called cosecant (csc), secant (sec) and cotangent (cot). Note that there are other functions, but beyond this recap.

There are other ways to express some of these relationships. For example: $\tan\theta = \frac{\sin\theta}{\cos\theta}$, or $\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta} = \frac{Hypotenuse}{Opposit}$, etc. The Pythagorean theorem can also be expressed in terms of trigonometric functions as: $\sin^2\theta + \cos^2\theta = 1$. Even using complex numbers, we can calculate the same functions. For example: $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$, or $\cos\theta = \frac{e^{i\theta} - e^{-i\theta}}{2}$, etc. However, we will only focus on basic formats.

Although we explained the meaning of the basic trigonometric functions using the right angle triangle, a trigonometric value of an angle is universal. For example, sine of a $30^0$ angle is always 0.5, or sin $90^0$=1.

As Excel fundamentally works in radians, we must be careful how we interpret the values if we try to calculate a sine of an angle of $30^0$ degrees, for example. If you entered =SIN(30), what you get is the value of -0.98803.., which is the sine value of an angle that is measured in radians, i.e. sine of 30 radians. If you want a sine of an angle that is 30 degrees, you must use a combination of Excel functions: =SIN(RADIANS(*number*)), where *number* is the value in degrees.

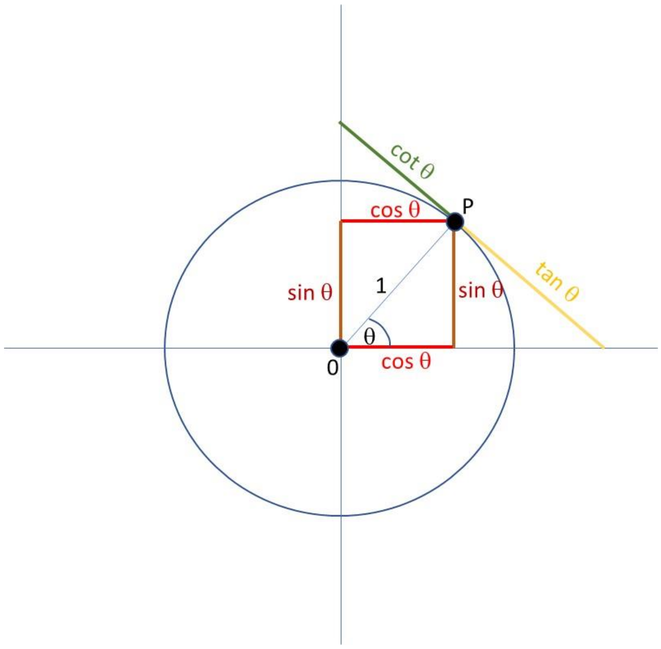
An example below shows incorrect value in cell B29 and the correct value for sine ($30^0$) in cell D29. The values in D30:D35 are the sine values of some of the more unique values of angles in degrees.

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 28 | Degrees | | | Sine | | |
| 29 | 30 | -0.988031624 | =SIN(A29) | 0.5 | =SIN(RADIANS(A29)) | |
| 30 | 45 | | | 0.707107 | | |
| 31 | 60 | | | 0.866025 | | |
| 32 | 90 | | | 1 | | |
| 33 | 180 | | | 1.23E-16 | | |
| 34 | 270 | | | -1 | | |
| 35 | 360 | | | -2.5E-16 | | |

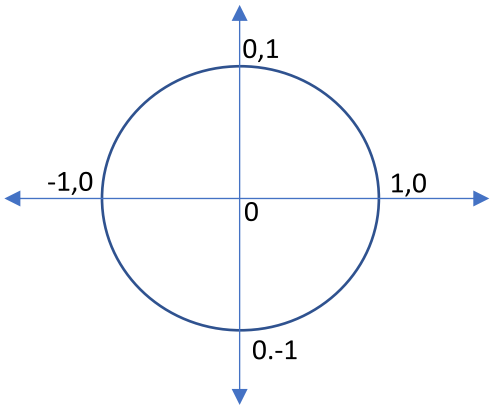
Let's return to the basic trigonometric functions. Below we show some of the angles and their corresponding trigonometric functions.

| | H | I | J | K | L | M | N | O | P | Q |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | **θ in degrees** | **0** | **30** | **45** | **60** | **90** | **135** | **180** | **270** | **360** |
| 2 | **θ in radians** | **0** | **π/6** | **π/4** | **π/3** | **π/2** | **3π/4** | **π** | **3π/2** | **2π** |
| 3 | sin θ | 0 | 1/2 | √2/2 | √3/2 | 1 | √2/2 | 0 | -1 | 0 |
| 4 | cos θ | 1 | √3/2 | √2/2 | 1/2 | 0 | -√2/2 | -1 | 0 | 1 |
| 5 | tan θ | 0 | √3/3 | 1 | √3 | - | -1 | 0 | - | 0 |
| 6 | cot θ | - | √3 | 1 | √3/3 | 0 | -1 | - | 0 | - |

To bring all these trigonometric functions in one pictorial, we could use the unit circle. This is the circle whose radius is equal to 1:

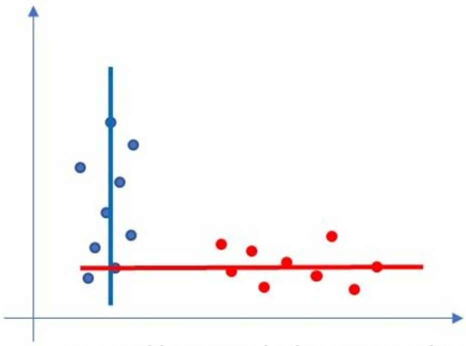



Let's put the coordinates at four prominent points coinciding with the sides of the world, such as North, South, East and West. Something like this:




If we were to show, for example, the sin value of an angle that has zero degrees, in other words, imagine an arrow going from 0 to (1,0), then sin of 0 is 0. Imagine now an arrow going from 0 to (0,1). This is now $90^0$ degrees in relation to 0 to (1,0), and the sine value of $90^0$ is 1. This implies that the sin of $180^0$ will be 0 and the sin of $270^0$ will be -1. When we come back to $360^0$, we have the sin of $360^0$ again equal to 0. So, clearly the sinus value is oscillating between 0 to 1, then again to 0 and to -1 before it returns back to 0.

The cosine values are opposite of sinus values. At the angle of zero, cosine is 1, at $90^0$ it is 0, etc. In other words, when the two points are on the two lines that are perpendicular to each other, the cosine value will be zero. This idea is very interesting in the context of the correlation coefficient. The closer the two variables are to each other, the closer the correlation coefficient to 1. If two variables have no connection, their coefficient is close to zero, and effectively they are perpendicular to one another.



Two variables perpendicular to one another. Angle between them $90^0$. Cos $90^0$ = 0 and Correlation coefficient = 0

Two variables moving in the same direction. Angle between them $0^0$. Cos $0^0$ = 1 and Correlation coefficient = 1

The above pictorials provide a graphical interpretation of the correlation coefficient and connects some aspects of trigonometry with statistics.

Let's close with a few more Excel functions. The sheet below contains number 30 in cells H9:H11. If we were to use sin, cos and tan function for these values, Excel would interpret that number 30 is given in radians. In this case, sin, cos and tan values in cells I9:I11 refer to an angle of 30 radians. To get proper values for the degrees (i.e. 30 degrees), as before, we need to convert radians into degrees and then calculate sin, cos and tan. This is given in cells K9:K11.

| | H | I | J | K | L | M |
|---|---|---|---|---|---|---|
| 8 | Value | Result from radians | | Result from degrees | | |
| 9 | 30 | -0.98803 | =SIN(H9) | 0.5 | =SIN(RADIANS(H9)) | |
| 10 | 30 | 0.154251 | =COS(H10) | 0.866025 | =COS(RADIANS(H10)) | |
| 11 | 30 | -6.40533 | =TAN(H11) | 0.57735 | =TAN(RADIANS(H11)) | |

The inverse functions in Excel are called arcsine =ASIN(number), arccosine =ACOS(number) and arctangent =ATAN(number). In other words, if we wanted to convert the sin, cos and tan values back into degrees (or radians), we would use one of these functions.

Cells H14:H16 below contain the numbers representing sin, cos and tan values. We would like to calculate the angle that corresponds to these values. Again, if we just used Excel function as in cells I14:I16, we get the values of an angle in radians. However, if we combine these functions with the function =DEGREES(number), we get the correct values in degrees. Cells K14:K16 show 30 degrees, which is the correct answer for the corresponding values for sin, cos and tan.

| | H | I | J | K | L | M |
|---|---|---|---|---|---|---|
| 13 | Value | Result to radians | | Result to degrees | | |
| 14 | 0.5 | 0.523599 | =ASIN(H14) | 30 | =DEGREES(ASIN(H14)) | |
| 15 | 0.86602 | 0.52361 | =ACOS(H15) | 30 | =DEGREES(ACOS(H15)) | |
| 16 | 0.57735 | 0.523599 | =ATAN(H16) | 30 | =DEGREES(ATAN(H16)) | |

To recap the most crucial point as how Excel treats different values, the table below uses examples of several angles between $0^0$ and $90^0$ and their sine values, as well as how to calculate the angles from the sine values.

| | H | I | J | K | L | M | N | O |
|---|---|---|---|---|---|---|---|---|
| 18 | Angle in degrees | 0 | 30 | 45 | 60 | 90 | | |
| 19 | Angle in radians | 0 | 0.523598776 | 0.785398163 | 1.047198 | 1.570796 | =RADIANS(M18) | |
| 20 | Sin from degrees | 0 | 0.5 | 0.707106781 | 0.866025 | 1 | =SIN(RADIANS(M18)) | |
| 21 | Sin from radians | 0 | 0.5 | 0.707106781 | 0.866025 | 1 | =SIN(M19) | |
| 22 | Angle in degrees from sin | 0 | 30 | 45 | 60 | 90 | =DEGREES(ASIN(M21)) | |
| 23 | Angle in radians from sin | 0 | 0.523598776 | 0.785398163 | 1.047198 | 1.570796 | =ASIN(M21) | |

## 2.11 Calculus

This refresher will cover only three topics from calculus, namely: the notion of limit, integrals and derivatives. These are the most fundamental concepts of calculus.

The concept of a limit is very intuitive. In calculus, a limit is the value that a function "approaches" as the input, or index, approaches some value. A limit is written as:

$$\lim_{n \to c} f(n) = L$$

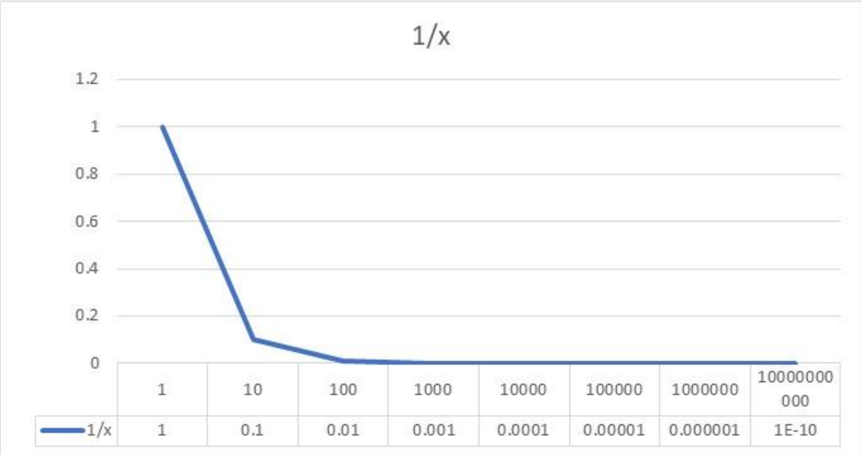
This is read as: "the limit of the function $f$ of n, as n approaches c, is equal to L. We already encounter this concept when we discussed compound interest. We said that as we increase the interval n for paying the interest, the formula $(1+r/n)^n$ was approaching the value of *e*, which is 2.718. This can be written as:

$$\lim_{n \to \infty} \left(1 + \frac{r}{n}\right)^n = e = 2.718$$

This expression "approaches certain value", such as infinity (or the alternative expression is "tends towards infinity"), means that this value (i.e. the infinity) will never be reached, but as we get closer, we begin to approach the limit L (which in our example above is 2.718).

We can take a simple function such as: $f(x) = \frac{1}{x}$. If we start with x=1 and increase towards infinity, we are effectively trying to find the limit for this function. Let's do a quick experiment:

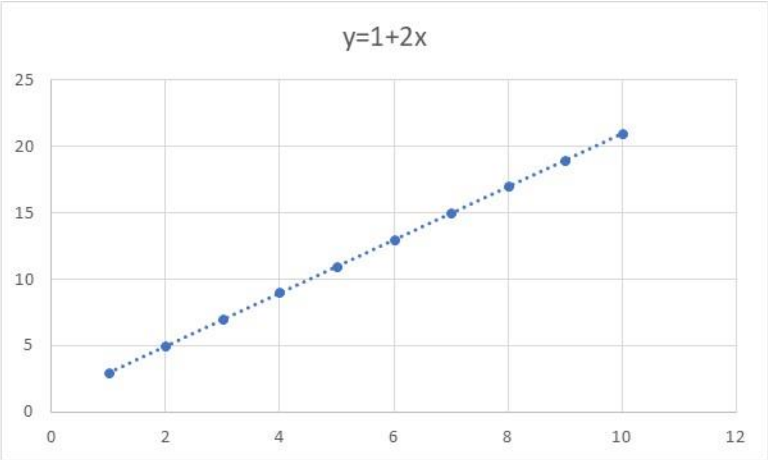
|   | A | B |
|---|---|---|
| 1 | x | 1/x |
| 2 | 1 | 1 |
| 3 | 10 | 0.1 |
| 4 | 100 | 0.01 |
| 5 | 1000 | 0.001 |
| 6 | 10000 | 0.0001 |
| 7 | 100000 | 0.00001 |
| 8 | 1000000 | 0.000001 |
| 9 | 10000000000 | 1E-10 |




As we can see, the larger the value of x, the closer the result is to zero. Or, as we said, if x tends towards infinity, the result approaches zero. In other words: $\lim_{n \to \infty} \frac{1}{x} = 0$. This is the concept of a limit.

Let's remind ourselves of the concept of the slope. Below we have a simple linear function y=1+2x. We can see that for every change in x, there is a change in y. In other words, the slope of the function is calculated as: $Slope = \frac{Change\ in\ y}{Change\ in\ x} = \frac{\Delta y}{\Delta x}$

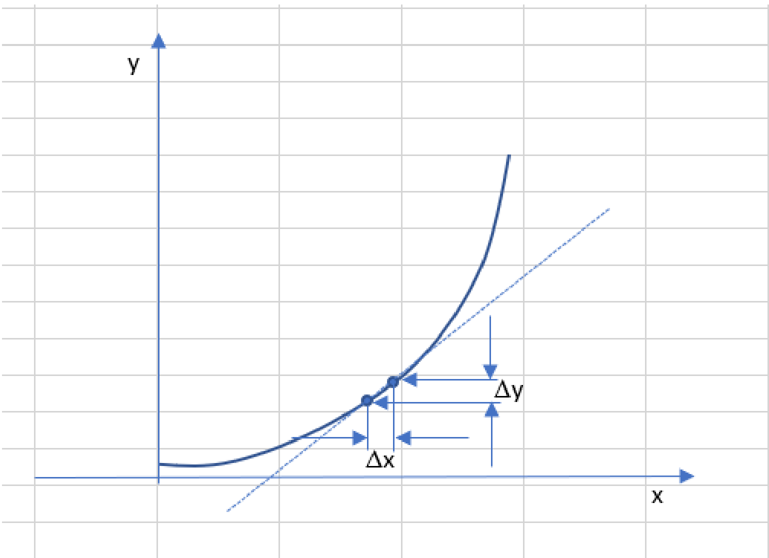
|   | P | Q |
|---|---|---|
| 1 | x | y=1+2x |
| 2 | 1 | 3 |
| 3 | 2 | 5 |
| 4 | 3 | 7 |
| 5 | 4 | 9 |
| 6 | 5 | 11 |
| 7 | 6 | 13 |
| 8 | 7 | 15 |
| 9 | 8 | 17 |
| 10 | 9 | 19 |
| 11 | 10 | 21 |




The symbol Δ (Greek letter delta) is used to represent the change. So, the slope is a result of the changes in y as the changes in x take place.

The above calculation is simple and logical for a simple reason that we have a function that is nothing but a straight line. What if the line is curve?

In this case, at any point of this curve, we can put a tangent, which is straight line. Let's now look at two dots on this tangent (straight) line that are so close together that they are both touching the curve. This is almost impossible, as the tangent can only touch a curve at one point, but imagine that these two points are so close that the distance between them is infinitesimally small (almost zero). In this case, we can use the same formula for the slope as we did earlier. A bit difficult to sketch this, but let's pretend that the two dots below are really, really close to one another.




We can take infinitesimally small changes in x and measure the changes in y. This will give us the slope of the curve at this point. What we are saying is that if x changes to x+Δx, and Δx is very small, then y, which is the function of x, i.e. $f$(x) will change to $f$(x+Δx).

Going back to our simple slope formula, we can say that:

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

The above formula should be rewritten as follows:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

We recognize this as the limit of the function from above. The formula reads: "The derivative of $f$ equals the limit of $f$(x+Δx)- $f$(x) over Δx, as Δx tends towards zero".

So clearly, the symbol for derivative is $f$'(x), or as it is sometimes written as dy/dx:

$$\frac{dy}{dx} = f'(x) = \frac{f(x + dx) - f(x)}{dx}$$

How do we put this into practise? Let's say that our function $f(x)=x^2$. How do we calculate the derivative of this function?

If $f(x)=x^2$, we want to see what is $f$(x+Δx). Well, we just need to take it to the power of 2, i.e.: $f(x+\Delta x)=(x+\Delta x)^2$, which can be written as $f(x+\Delta x)=x^2 + 2x\ \Delta x +( \Delta x)^2$. To calculate the slope, we have:

$$slope = \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} = \frac{2x\Delta x + (\Delta x)^2}{\Delta x} = 2x + \Delta x = 2x$$

In the above equation, at the very end, we just got rid of Δx, because it infinitesimally small and virtual zero, so we can treat it as non-existent in the equation. The answer is that if we have $f(x)=x^2$, then the derivative of this function is $f$'(x)=2x. In fact, there are numerous rules for calculating derivatives, but we will not go into this. We just want to understand the meaning of the derivative of a function.

From what we saw so far, derivatives will also create a function, so if the original function was non-linear, a parabola for example, such as $f(x)=x^2$, the first derivative of this function is a linear function $f$'(x)=2x.

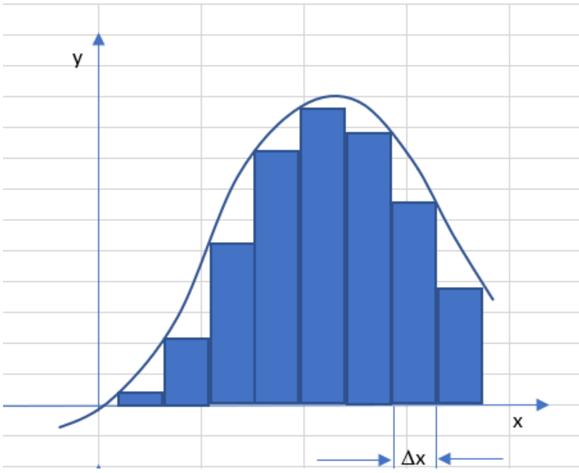
Essentially, the derivative is the rate of change of a function at any given point. So, the derivative is nothing but the rate of change of the function. To use the real-life analogy, and let's take driving a car example, then derivative is the same as the velocity at which this car moves at any point in time x. If it is positive, it increases, if it is negative, it decreases.

You can also take the second derivative of the function too, which is symbolised as $f$''(x). The second derivative is, in the case of the car example, equivalent to the acceleration. Sounds much simpler than saying the rate of change of the rate of change.

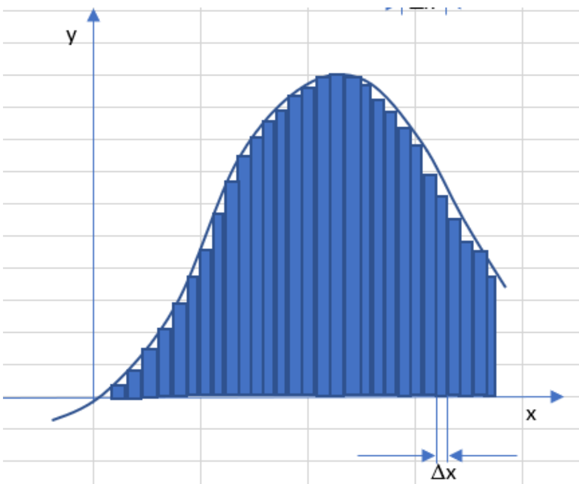
If we were to use analogy from business and economics, then finding a derivative of the cost function, for example, is called the marginal cost. A derivative of the revenue function, for example, is called the marginal revenue. In both case the interpretation is something as follows: if the total costs for producing x number of units is described by a function $f$(x), then $f$'(x), or the derivative of this function, gives you the value for producing just one more incremental unit. The same goes for the revenues.

Another way to think about derivatives is to consider a function that describes the price level of an economy at any point in time. The first derivative of this function is the inflation rate (changes in price level), and the second derivative shows the rate at which the inflation is changing.

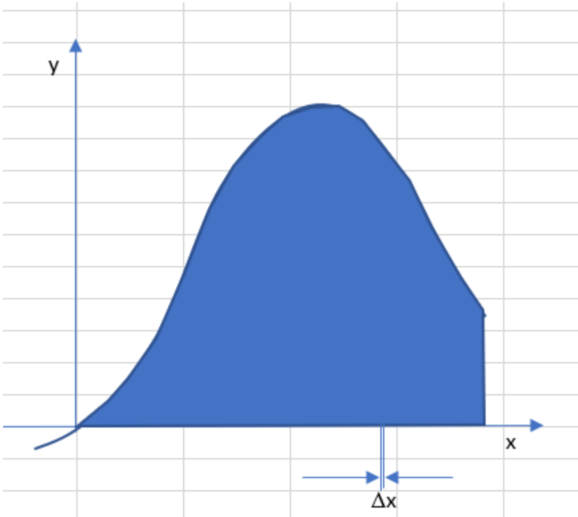
Integration is an opposite operation of derivation. If $f'(x)=2x$, then after we calculated the integral of this function, which is $\int$, the answer is: $\int 2x\,dx = x^2 + c$. As we can see, we are back to our original function $f(x)=x^2$, except we have some "dangling" letter c. Let's see what is the meaning of this c.




The second picture illustrates this scenario. It is still to imprecise. What if we take Δx to be very, very small?




Then, as the third picture shows, we would get close to the true area under the curve.




Imagine we have a function $f$(x), which is a curve as depicted above. Let's try to calculate the area under this curve. The first graph shows one possible option. We put a series of bars whose width is $\Delta x$ and we calculate the area for all these bars. After we add all these areas up, we have an approximate area below the curve. Clearly these bars are too wide, so we need our $\Delta x$ to be smaller. The second picture illustrates this scenario. It is still to imprecise. What if we take $\Delta x$ to be very, very small? Then, as the third picture shows, we would get close to the true area under the curve.

As before, this is the limit game. The smaller $\Delta x$ is, or, as it tends towards zero, the more precise our area calculation will be, because it approaches the limit of this function.

As before with derivatives, integrals have their rules, and we will not go into any of the details here. However, after integrating, for example, $\int 2x\,dx = x^2 + c$, the first part of our result is 2x. Because of the rules of integration, there are many functions that will have the first part of the answer as $x^2$. For example, $x^2+1$, or $x^2+12$, or $x^2-2$, all have the first part of the result as $x^2$. However, the derivative of a constant is zero. In other words, the numbers 1, 12 or -2 in our previous examples will all produce zero for a derivative. To reverse the operation, we say that if the derivative of a constant is zero, then the integral should have the constant attached to the first part of the answer. Hence, c at the end. In other words, we do not know what the constant is, but we know that there could be one after we integrate a function.

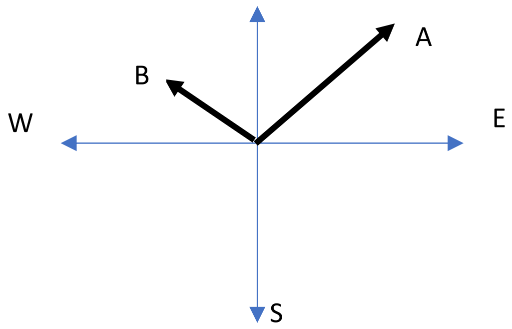
What is the interpretation of integrals in business and economics? It is opposite of derivatives, i.e. they are used to find the total cost function, the total revenue function, etc. A typical engineering interpretation of an integral vary from how much asphalt to fill in a hole in civil engineering, to calculating circuits in electronics. Calculating an average value of a curve is also done using integrals.

## 2.12 Vectors

Vector represents a quantity that has a magnitude and direction. Velocity is a good example of a vector. Velocity is not just a speed. It is a speed **and** a direction.

Vectors can be easily presented in a graphical form. Below are two vectors. One shows a car A traveling at 60 mph in the NE direction and the other one a car B traveling at 30 mph in the NW direction.

N




Vectors are represented with bold small letters, such as **a** or **b**. The magnitude of the vector is represented as Modulus **a**, or Mod **a**. This is also shown as $|\mathbf{a}|$ or $\|\boldsymbol{a}\|$.

Vectors can be added, subtracted, multiplied, etc. Most of the general rules of algebra when dealing with ordinary numbers apply equally to vectors. For example, **a** + **b** = **b** + **a**, or **a** + (**b** + **c**) = (**a** + **b**) + **c**, or **a** × 0 = 0, etc.

In algebra, vectors are represented as a single column of numbers in square or round brackets. Below are two examples of vectors:

$$\begin{pmatrix} -2 \\ 3 \end{pmatrix} \text{ or } \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

To **add two vectors**, we simply add the corresponding elements from each vector:

$$\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 0 \end{bmatrix}$$

If the vectors are not of the same size, you cannot add them.

Scalars are also quantities, but unlike vectors, they only have the magnitude (not direction). A good example of a scalar is length, or power, or volume, etc. Vectors can be multiplied by scalars. Scalars are represented by a single number. To multiply a vector **a** with scalar 3, for example, we get 3**a**.

As before, most of general rules of algebra (commutative law, associative law, distributive, etc.) apply to vectors and scalars. For example, m**a** = **a**m, or m(**a**+**b**) = m**a** + m**b**, etc.

To execute addition or subtraction in Excel, we do not have any dedicated functions, but we use the array data entry, which means a collection of cells is entered with CTRL, ALT and ENTER keys pressed simultaneously. Below are the two examples:

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | **Vectors** | | | | | |
| 2 | Adding and subtracting vectors | | | | | |
| 3 | 2 | 3 | | 5 | {=A3:A5+B3:B5} | |
| 4 | 1 | 5 | | 6 | {=A3:A5+B3:B5} | |
| 5 | 8 | 9 | | 17 | {=A3:A5+B3:B5} | |
| 6 | | | | | | |
| 7 | 2 | 3 | | -1 | {=A7:A9-B7:B9} | |
| 8 | 1 | 5 | | -4 | {=A7:A9-B7:B9} | |
| 9 | 8 | 9 | | -1 | {=A7:A9-B7:B9} | |

The first example is adding two vectors, one in cells A3:A5 and the other one in B3:B5. We first highlight the cells D3:D5, where the result will be placed. After that, we just type A3:A5+B3:B5. However, we do not press ENTER. We press CTRL, ALT and ENTER key simultaneously. This marks cells D3:D5 as an array and produces the correct results. The curly brackets are added automatically by Excel to imply that this is an array. The same applies to subtraction, as shown in cells A7:D9.

**Multiplication and division** are executed in Excel exactly in the same way. See the example below.

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 11 | Multplying vectors | | | | | |
| 12 | 2 | 3 | | 6 | {=A12:A14*B12:B14} | |
| 13 | 1 | 5 | | 5 | {=A12:A14*B12:B14} | |
| 14 | 8 | 9 | | 72 | {=A12:A14*B12:B14} | |
| 15 | | | | | | |
| 16 | 2 | 3 | | 0.666667 | {=A16:A18/B16:B18} | |
| 17 | 1 | 5 | | 0.2 | {=A16:A18/B16:B18} | |
| 18 | 8 | 9 | | 0.888889 | {=A16:A18/B16:B18} | |

Let's say we want to multiply each vector above with a scalar m, where m=2, and let's use the same vectors as above. The two vectors become:

$$2 \times \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} \qquad 2 \times \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 10 \end{bmatrix}$$

Clearly, to **multiply a vector with a scalar** m, in the first case we used $\begin{bmatrix} -2m \\ 3m \end{bmatrix}$ and in the second $\begin{bmatrix} 3m \\ 2m \\ 5m \end{bmatrix}$.

In Excel, to multiply a vector with a scalar, we use the same technique as above. The example below demonstrates it.

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 20 | Multiply with a scalar | | | | | |
| 21 | 2 | | | 18 | {=A21:A23*B22} | |
| 22 | 1 | 9 | | 9 | {=A21:A23*B22} | |
| 23 | 8 | | | 72 | {=A21:A23*B22} | |

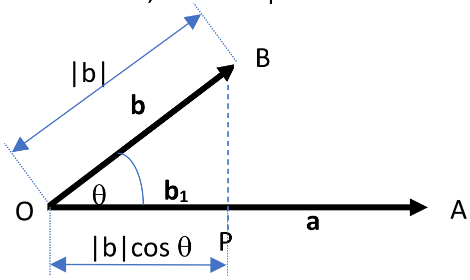
To calculate the **magnitude of a vector**, which as we said is symbolised as $|\mathbf{a}|$, you need to square all the elements in the vector and then take the square root. For example:

$$|\mathbf{a}| = \begin{bmatrix} 2 \\ -4 \\ 2 \\ 1 \end{bmatrix} = \sqrt{2^2 + (-4)^2 + 2^2 + 1^2} = \sqrt{25} = 5$$

To do the same in Excel, we combine the functions =SQRT() and =SUMSQ(), as shown below:

| | H | I | J | K | L | M | N |
|---|---|---|---|---|---|---|---|
| 2 | Magnitude | | | | | | |
| 3 | 2 | | | | | | |
| 4 | -4 | | | 5 | =SQRT(SUMSQ(H3:H6)) | | |
| 5 | 2 | | | | | | |
| 6 | 1 | | | | | | |

**Scalar product** of two vectors, for example vectors **a** and **b**, can be visualised as follows:




The projection of vector **b** on vector **a** is represented as $\mathbf{b_1}$. The scalar product of the vectors **a** and **b** is then: **a • b** = |**a**| |$\mathbf{b_1}$|. The scalar product is also called the **dot product**. From the above sketch we can see that θ is the angle between vectors **a** and **b**. This means that the relationship can be expressed as |$\mathbf{b_1}$| = |**b**| cos θ. Equally, the above expression for the dot product can also be shown as: **a • b** = |**a**| |**b**|cos θ.

To calculate a scalar product between two vectors in Excel, we use function =SUMPRODUCT(range1, range2). Below is an example.

| | H | I | J | K | L | M | N |
|---|---|---|---|---|---|---|---|
| 8 | Scalar product | | | | | | |
| 9 | 2 | 3 | | | | | |
| 10 | 1 | 5 | | 83 | =SUMPRODUCT(H9:H11,I9:I11) | | |
| 11 | 8 | 9 | | | | | |

As we can see in cell K10, there is no need to use the array formula entry as the dot product is a single number.

When two vectors are at right angles at each other, the dot product is equal to zero.

## 2.13 Matrices

What if we have more columns in a vector, such as: $\begin{pmatrix} 2 & 7 \\ -1 & 3 \end{pmatrix}$? Well, then they are not called vectors, but matrices. In fact, a single column matrix is a vector.

Just like vectors, in algebra matrices are expressed in bold letters, but as capital letters. For example:

$$\mathbf{A} = \begin{pmatrix} 15 & 3 & 2 \\ 17 & 2 & 5 \\ 32 & 5 & 7 \end{pmatrix}$$

This means that, just like with vectors, majority of the usual laws of algebra apply to matrices. So, for example: **A** + **B** = **B** + **A**, or **A** + (**B** + **C**) = (**A** + **B**) + **C**, etc. Multiplying with a scalar (single number), is also identical to vectors and the usual laws of algebra: m**a** = **a**m, or m(**A** + **B**) = m**A** + m**B**, etc.

Any table can be represented as a matrix. For example, let's look at a mix of nationalities in an adult learning class:

| | English | Scottish | Irish | Welsh |
|---|---|---|---|---|
| Men | 15 | 3 | 2 | 1 |
| Women | 17 | 2 | 5 | 3 |

Table: Nationalities in a class per gender

This table, if represented as a rectangular array of numbers, becomes a matrix:

$$\begin{pmatrix} 15 & 3 & 2 & 1 \\ 17 & 2 & 5 & 3 \end{pmatrix}$$

In the above table, we might like to get the totals per columns and rows. The table then looks as:

| | English | Scottish | Irish | Welsh | Total |
|---|---|---|---|---|---|
| Men | 15 | 3 | 2 | 2 | 22 |
| Women | 17 | 2 | 5 | 3 | 27 |
| Total | 32 | 5 | 7 | 4 | 49 |

Table: Nationalities in a class per gender

Let's look at the column with the totals for men and women. We have effectively added:

$$\begin{pmatrix} 15 \\ 17 \\ 32 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} + \begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 22 \\ 27 \\ 49 \end{pmatrix}$$

This means that a matrix can be treated as a series of vectors, and by adding all the individual elements of every vector, we get the sum of the matrix.

To add the numbers from B3:B5, C3:C5, D3:D5 and E3:E5 in Excel and put the result in H3:H5 (which is identical to the result in F3:F5 achieved by a simple =SUM() function), we used the same technique as with vectors and applied the array formula.

| | A | B | C | D | E | F | G | H | I | J | K |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | **Matrices** | | | | | | | | | | |
| 2 | | English | Scottish | Irish | Welsh | Total | | | | | |
| 3 | Men | 15 | 3 | 2 | 2 | 22 | | 22 | {=B3:B5+C3:C5+D3:D5+E3:E5} | | |
| 4 | Women | 17 | 2 | 5 | 3 | 27 | | 27 | {=B3:B5+C3:C5+D3:D5+E3:E5} | | |
| 5 | Total | 32 | 5 | 7 | 5 | 49 | | 49 | {=B3:B5+C3:C5+D3:D5+E3:E5} | | |

Sometimes it is convenient to assign index numbers to the elements in a matrix, to make sure they are more easily identified. For example:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

The above notation also indicates that matrices have dimensions. The dimension (or size) of the above matrix is 3×3. However, the dimension does not have to be square, such as n×n, it can be n×m.

To **add and subtract matrices**, assuming they are square, means that we need to add/subtract the corresponding elements from the two matrices. Here is an example:

$$\begin{pmatrix} 1 & 3 \\ 5 & -4 \end{pmatrix} + \begin{pmatrix} 6 & 3 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} 7 & 6 \\ 3 & -5 \end{pmatrix}$$

To add and subtract matrices in Excel, we use the same techniques as adding and subtracting vectors. Example below demonstrates it.

| | A | B | C | D | E | F | G | H | I | J | K |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 7 | Adding and subtracting matrices | | | | | | | | | | |
| 8 | 1 | 3 | | 6 | 3 | | 7 | 6 | {=A8:B9+D8:E9} | {=A7:B8+D7:E8} | |
| 9 | 5 | -4 | | -2 | -1 | | 3 | -5 | {=A8:B9+D8:E9} | {=A7:B8+D7:E8} | |

We again highlighted four cells, which will be the matrix with the results (G8:H9). In this case, we added cells A8:B9, which was one matrix, with D8:E9, which was the second matrix. After we typed the formula, we entered it as an array formula pressing CTRL, ALT and ENTER simultaneously.

If we **multiply two matrices** that have the same number of rows and columns (square matrices), the resulting matrix will have the identical number of rows and columns. The multiplication rule is:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

For multiplying two matrices, we use Excel function =MMULT(array1, array2). The example is given below. This is also an array formula.

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| 11 | Multiplying matrices | | | | | | | | | |
| 12 | 1 | 3 | | 6 | 3 | | 0 | 0 | {=MMULT(A12:B13,D12:E13)} | |
| 13 | 5 | -4 | | -2 | -1 | | 38 | 19 | | |

The formula {=MMULT(A12:B13,D12:E13) applies to all four cells G12:H13. As before, the cells were first highlighted, after which we type this formula and press ALT, CTRL and ENTER simultaneously.

We already know how to multiply the vector (which is a single column matrix) with a scalar. The same applies when we want to **multiply a matrix with a scalar**:

$$5 \times \begin{pmatrix} 2 & -3 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 10 & -15 \\ 0 & 20 \end{pmatrix}$$

In Excel multiplication of a matrix with a scalar is executed the same way as multiplication of a vector with a scalar. Below is an example.

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 16 | 2 | -3 | | | 10 | -15 | {=A16:B17*C17} | |
| 17 | 0 | 4 | 5 | | 0 | 20 | | |

As we can see, this is the same array formula used for vector and scalar example.

If we multiply two matrices that do not have the same number of rows and columns, the resulting matrix will be as in the example below:

$$\begin{pmatrix} 15 & 3 & 2 \\ 17 & 2 & 5 \\ 32 & 5 & 7 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 3 & -1 \\ 5 & 0 \end{pmatrix} = \begin{pmatrix} 79 & 12 \\ 99 & 15 \\ 178 & 27 \end{pmatrix}$$

As above, the multiplication is achieved always by multiplying the row in the first matrix by the column in the second matrix. So, to get 79, we do the following: (15×4) + (3×3) + (2×5) = 79. To get 15 in the second row, second column, for example, we do: (17×1) + (2×-1) + (5×0) = 15, etc.

Or in more general terms:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} = \begin{pmatrix} (a_{11} \times b_{11}) + (a_{12} \times b_{21}) + (a_{13} \times b_{31}) & (a_{11} \times b_{12}) + (a_{12} \times b_{22}) + (a_{13} \times b_{32}) \\ (a_{21} \times b_{11}) + (a_{22} \times b_{21}) + (a_{23} \times b_{31}) & (a_{21} \times b_{12}) + (a_{22} \times b_{22}) + (a_{23} \times b_{32}) \\ (a_{31} \times b_{11}) + (a_{32} \times b_{21}) + (a_{33} \times b_{31}) & (a_{31} \times b_{12}) + (a_{32} \times b_{22}) + (a_{33} \times b_{32}) \end{pmatrix}$$

Another example, though the same rules apply is:

$$\begin{pmatrix} 6 & 2 & 9 \\ 3 & 5 & 0 \end{pmatrix} \begin{pmatrix} 2 & 7 \\ 1 & -2 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} 50 & 38 \\ 11 & 11 \end{pmatrix}$$

We got the result by applying the previous rule:

$$\begin{bmatrix} 6\times 2 + 2\times 1 + 9\times 4 & 6\times 7 + 2\times(-2) + 9\times 0 \\ 3\times 2 + 5\times 1 + 0\times 4 & 3\times 7 + 5\times(-2) + 0\times 0 \end{bmatrix}$$

In Excel, we use the same function =MMULT(array1, array2) to multiply any size of a matrix. Example below shows it.

| | A | B | C | D | E | F | G | H | I | J | K | L |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 19 | Multiplying matrices 3x2 or 2x3 | | | | | | | | | | | |
| 20 | 15 | 3 | 2 | | 4 | 1 | | 79 | 12 | {=MMULT(A20:C22,E20:F22)} | | |
| 21 | 17 | 2 | 5 | | 3 | -1 | | 99 | 15 | | | |
| 22 | 32 | 5 | 7 | | 5 | 0 | | 178 | 27 | | | |
| 23 | | | | | | | | | | | | |
| 24 | 6 | 2 | 9 | | 2 | 7 | | 50 | 38 | {=MMULT(A24:C25,E24:F26)} | | |
| 25 | 3 | 5 | 0 | | 1 | -2 | | 11 | 11 | | | |

The only difference from square matrices or vectors is how many cells we are going to highlight before we enter the array formula.

What is a **transposed matrix**, i.e. $\mathbf{A}^T$or **A'**? If we change rows into columns, we get a transposed matrix:

**A** = $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ becomes $\mathbf{A}^T$ or **A'** = $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

A more general expression is: **A**=$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ becomes $\mathbf{A}^T$=$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$, or $(a_{ij})^T=a_{ji}$.

To transpose a matrix in Excel, we used a dedicated function =TRANSPOSE(array). This is an array function and the example below shows how to use it.

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 27 | Matrix | | | Transposing matrices | | | | |
| 28 | 1 | 2 | | 1 | 3 | {=TRANSPOSE(A28:B29)} | | |
| 29 | 3 | 4 | | 2 | 4 | | | |

As before, we first highlight the area where the transposed matrix will be, in our case D28:E29, then enter the function =TRANSPOSE(A28:B29) and press simultaneously CTRL, ALT and ENTER.

If we have a square matrix (the same number of rows and columns), then we can calculate a **determinant of a matrix**. Determinant of a matrix is denoted as det **A**, or just as: det = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. To find the determinant of **A**, we follow the rule: If **A** = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then det **A** =$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ = ad -bc

For example: If **A** = $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, then det **A** = $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$= 1×4 – 3×2 = -2

For a 3×3 matrix, a determinant is calculated as follows:

**A** = $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, then det **A** =$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ = aei -ahf+dhc-dbi+gbf-gec

Excel function for determinants is =MDETERM(array) and the example below shows how to use it.

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 31 | Determinants | | | | | | |
| 32 | 2 | 3 | 4 | | -40 | =MDETERM(A32:C34) | |
| 33 | 1 | 5 | 7 | | | | |
| 34 | 8 | 9 | 6 | | | | |

For larger matrices, manual calculations are just a bit more complex, as they require understanding how to find the matrix cofactor. However, in Excel the function =MDTERM(array) does not require any deeper knowledge of matrix algebra. In any case, remember that only square matrices can have a determinant.

There are a few unique matrices, one of them is called a **zero matrix**. Here is an example of a 2×2 zero matrix: $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Another unique type of a square matrix is **identity matrix**, or sometimes called the **unity matrix**. A 2×2 identity matrix looks as follows: **I** = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. A 3×3 identity matrix is: **I** = $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Both zero and identity matrices are necessary so that the laws of algebra can be followed, hence the term matrix algebra. Most of the laws that apply to numbers, apply to matrices, with few exceptions.

A matrix can also be inversed, so **A** can be turned into **A**$^{-1}$. When the original matrix is multiplied by its **<u>inverse matrix</u>**, the result is always the identity matrix. For example: **A A**$^{-1}$ = **A**$^{-1}$ **A** = **I**. However, this is only possible if det **A** is not equal to zero (i.e. det **A** $\neq$ 0).

It can also be said that if **A** = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then **A**$^{-1}$ = $\frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Let's use an example.

**A** = $\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$, and **det A** = $\begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix}$ = 3×4 - 2×1 = 10, then **A**$^{-1}$ = $\frac{1}{10}\begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 0.4 & -0.2 \\ -0.1 & 0.3 \end{pmatrix}$.

In Excel, we can use function =MINVERSE(array) to calculate the inverse of the matrix. The example below demonstrates it.

V23

| | N | O | P | Q | R | S | T |
|---|---|---|---|---|---|---|---|
| 2 | Matrix | | | Inverse matrix | | | |
| 3 | 3 | 2 | | 0.4 | -0.2 | {=MINVERSE(N3:O4)} | |
| 4 | 1 | 4 | | -0.1 | 0.3 | | |

As before, =MINVERSE(array) is an array formula, so it is used like all other array formulas.

We said that **A A**$^{-1}$ = **I**. We can verify this on our practical example:

**A** × **A**$^{-1}$ = $\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} \times \frac{1}{10}\begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

In Excel, we can achieve this effortlessly, as per the example below.

| | N | O | P | Q | R | S | T | U | V | W | X | Y | Z |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | Matrix | | | Inverse matrix | | | | | Multiply matrix with its inverse | | | | |
| 3 | 3 | 2 | | 0.4 | -0.2 | {=MINVERSE(N3:O4)} | | | 1 | 0 | {=MMULT(N3:O4,Q3:R4)} | | |
| 4 | 1 | 4 | | -0.1 | 0.3 | | | | 0 | 1 | {=MMULT(N3:O4,Q3:R4)} | | |

We used function =MMULT(array1, array2) to multiply two matrices. The first matrix is N3:O4 and its inverse is given in Q3:R4. The function is an array function, so the result is another matrix in V3:W4, and as we can see it is the unity matrix, as expected. Again, remember that if a matrix does not have a determinant, i.e. it is equal to zero, then you cannot find the inverse of the matrix.

To **<u>divide two matrices</u>** is quite a complex procedure, involving most of the terms we covered so far. Fortunately, in Excel it could not be simpler. If we have two matrices **A** and **B**, then **A** ÷ **B** = **A** × **B**$^{-1}$. This means that to divide **A** with **B**, all we have to do is to multiply **A** with the inverse of **B**, i.e. **B**$^{-1}$. Below is the example from Excel.

| | N | O | P | Q | R | S | T | U | V | W | X | Y |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 6 | Divide matrices | | | | | | | | | | | |
| 7 | 13 | 26 | | 7 | 4 | | -1 | 10 | {=MMULT(N7:O8,MINVERSE(Q7:R8))} | | | |
| 8 | 39 | 13 | | 2 | 3 | | 7 | -5 | | | | |

Matrix **A**, in our case, is in N7:O8 and matrix **B** is in Q7:R8. We used again array function =MMULT(), but this time we combined it with the function =MINVERSE(). As it is an array formula, we first highlighted the area where the result will be, i.e. T7:U8, then entered the formula =MMULT(N7:O8,MINVERSE(Q7:R8)) and pressed CTRL, ALT and ENTER. The result is shown.

If inverse matrix (**B**$^{-1}$ in our case**)** does not exist because **B** has no inverse (since det **B** = 0), you cannot calculate this in Excel.

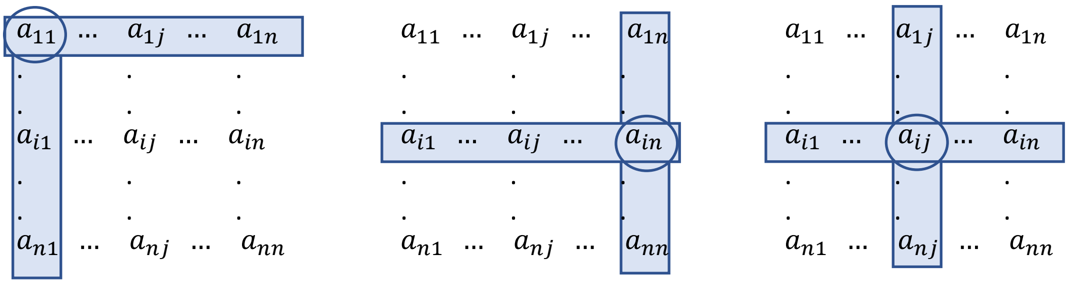
It is extremely easy to use the function =MINVERSE() in Excel. However, if we had to do it manually, the procedure would be very taxing. Let's explain. The next section might be a bit challenging, so be patient. If you can digest the first few paragraphs, you will gain full understanding of the complexity of calculating the inverse of the matrix.

Let **C**$^T$ be the transpose matrix of **cofactors** of matrix **A**. This transposed matrix of cofactors **C**$^T$ is called the **comatrix** of **A**, or the adjoint of **A**. Sometimes the adjoint is written not as **C**$^T$, but as cof **A,** or adj **A**. To calculate the transposed matrix of cofactors, we start with the equation:

$\mathbf{A}^{-1} = \frac{1}{\det A}\boldsymbol{C}^T$ or $\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\mathrm{com}\,\mathbf{A}$

From there: $\boldsymbol{C}^T = \mathbf{A}^{-1} \det \boldsymbol{A}$ or $\mathrm{cof}\,\mathbf{A} = \mathbf{A}^{-1} \det \boldsymbol{A}$

So, what are the cofactors? If we take a square matrix, then for every element in the matrix we can calculate the cofactor. To calculate the cofactors, we first need to create a series of **matrix minors**. A minor of the matrix is created by eliminating the row and the column for every element of the matrix. For example, to calculate the cofactor for element $a_{11}$ below, we create a minor by eliminating all the elements from row 1 and column 1.



In the second example above, we want to calculate the cofactor for element $a_{in}$. This means eliminating the i$^{th}$ row and the n$^{th}$ column to create a minor. In the third example, $a_{ij}$ element's cofactor is calculated by eliminating the i$^{th}$ row and the j$^{th}$ column to create a minor.

Effectively, we "resize" a matrix **A** by removing one or more of its rows or columns. So, why do we need minors? They are used for calculating matrix cofactors, and cofactors are useful for computing both the determinant and inverse of a square matrix. Let's go back to the calculations.

Once we have a series of minors, which is a series of submatrices, we need to calculate the determinant for every minor matrix. A series of these determinants will become a new matrix of the cofactors.

If we say that a series of minors of a matrix **A** are called $M_{ij}$, and cofactors are called $A_{ij}$, then the relationship between the two is:

$$(-1)^{i+j} \times M_{ij} = A_{ij}$$

Sounds complicated, but it isn't. It is just a bit convoluted.

But how are the cofactors calculated from minors? They are calculated in the same way as the determinant. So, for example, if we had a 3×3 matrix and we wanted to calculate just a few cofactors to illustrate the principles, here is what we would do.

For a matrix $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 7 & 6 \end{pmatrix}$, to calculate the cofactor $A_{13}$, we would do the following:

$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 7 & 6 \end{pmatrix}$. The minor is $\begin{pmatrix} 0 & 4 \\ 1 & 7 \end{pmatrix}$. Now calculate the determinant of $\begin{vmatrix} 0 & 4 \\ 1 & 7 \end{vmatrix}$ = (0×7)-(1×4) = -4

To calculate the cofactor for $A_{22}$, for example:

$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 7 & 6 \end{pmatrix}$. The minor is $\begin{pmatrix} 1 & 3 \\ 1 & 6 \end{pmatrix}$. Now calculate the determinant of $\begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix}$ = (1×6)-(3×1) = 3

More formally, we say that a cofactor of $a_{ij}$ in **A** is the scalar, whose value is:

$$\text{cof } a_{ij} = (-1)^{i+j} \det A_{ij}$$

Or, this can be put differently as:

$$\det A = \sum_{i=1}^{n} a_{ij}\, cof\ a_{ij}$$

You can say that a cofactor of an element in n × n matrix is effectively an (n-1) × (n-1) determinant.

Now we understand the minors and cofactors, we can calculate the inverse of the matrix. The conventional procedure is quite convoluted.

Let's say we have a 3×3 matrix: $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 7 & 6 \end{pmatrix}$

The normal procedure is:

1. Extract the series of minors and calculate the determinant for every minor matrix:

$$A_{11} = \begin{vmatrix} 4 & 5 \\ 7 & 6 \end{vmatrix} = -11 \qquad A_{12} = \begin{vmatrix} 0 & 5 \\ 1 & 6 \end{vmatrix} = -5 \qquad A_{13} = \begin{vmatrix} 0 & 4 \\ 1 & 7 \end{vmatrix} = -4$$

$$A_{21} = \begin{vmatrix} 2 & 3 \\ 7 & 6 \end{vmatrix} = -9 \qquad A_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} = 3 \qquad A_{23} = \begin{vmatrix} 1 & 2 \\ 1 & 7 \end{vmatrix} = 5$$

$$A_{31} = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = -2 \qquad A_{32} = \begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} = 5 \qquad A_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} = 4$$

2. Fill the cofactor matrix with the determinant values and change the signs:

$$\boldsymbol{C} = \begin{pmatrix} -11 & -5 & -4 \\ -9 & 3 & 5 \\ -2 & 5 & 4 \end{pmatrix} \times \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} = \begin{pmatrix} -11 & 5 & -4 \\ 9 & 3 & -5 \\ -2 & -5 & 4 \end{pmatrix}$$

3. Calculate the adjugate (or adjoint) of matrix **A**, i.e. $\mathbf{C}^T$, which is the transpose of the cofactor matrix **C**:

$$\boldsymbol{C}^T = \begin{pmatrix} -11 & -9 & -2 \\ 5 & 3 & -5 \\ -4 & -5 & 4 \end{pmatrix}$$

4. Calculate the determinant of the initial matrix **A**.

$$det\ \boldsymbol{A} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 7 & 6 \end{vmatrix} = [(-11\times3\times4)+(2\times5\times1)+(3\times0\times7)]-[(3\times4\times1)+(2\times0\times6)+(1\times5\times7) = -13$$

5. Now multiply the adjugate with the reciprocal of the determinant:

$$\boldsymbol{A}^{-1} = \frac{1}{\det A}\text{adjoint}\ \mathbf{A} = \frac{1}{\det A}C^T$$

$$\boldsymbol{A}^{-1} = \frac{1}{-13}\begin{pmatrix} -11 & -9 & -2 \\ 5 & 3 & -5 \\ -4 & -5 & 4 \end{pmatrix} = \begin{pmatrix} 0.85 & -0.69 & 0.15 \\ -0.38 & -0.23 & 0.38 \\ 0.31 & 0.38 & -0.31 \end{pmatrix}$$

Now you can fully appreciate the power of Excel. The same example calculated in Excel using the Excel function =MINVERSE() is almost trivial. Here is the same example, using the same matrix, but in Excel:

| | N | O | P | Q | R | S | T | U | V | W |
|---|---|---|---|---|---|---|---|---|---|---|
| 10 | Matrix A | | | | | Inverse of A using Excel | | | | |
| 11 | 1 | 2 | 3 | | | 0.846154 | -0.69231 | 0.153846 | {=MINVERSE(N11:P13)} | |
| 12 | 0 | 4 | 5 | | | -0.38462 | -0.23077 | 0.384615 | | |
| 13 | 1 | 7 | 6 | | | 0.307692 | 0.384615 | -0.30769 | | |

As we can see, we do not even have to know what minors and cofactors are, and the result is returned.

One advantage of using matrices is that they could be used to solve **simultaneous equations**. For example, a set of simultaneous equations is:

ax + by = p
cx + dy = q

Let's assign elements from this system of simultaneous equations into matrices. We get:

$$\boldsymbol{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\ \boldsymbol{N} = \begin{pmatrix} x \\ y \end{pmatrix}\ \boldsymbol{R} = \begin{pmatrix} p \\ q \end{pmatrix}$$

This means that **MN** = **R**, or: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$

The multiplication rule from above takes us back to the original form: $\begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$

If we have **MN** = **R**, or: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$, then we can multiply both side by $\mathbf{M}^{-1}$:

$\mathbf{M}^{-1}\mathbf{MN} = \mathbf{M}^{-1}\mathbf{R}$, this yields **IN** = $\mathbf{M}^{-1}\mathbf{R}$ and ultimately **N** = $\mathbf{M}^{-1}\mathbf{R}$. Let's see an example.

3x + 6y = 21
2x + 5y = 16 or

$$\begin{pmatrix} 3 & 6 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 21 \\ 16 \end{pmatrix}$$

Thus

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 2 & 5 \end{pmatrix}^{-1}\begin{pmatrix} 21 \\ 16 \end{pmatrix}$$

The determinant of **M** is:

$$\mathbf{det\ M} = \begin{vmatrix} 3 & 6 \\ 2 & 5 \end{vmatrix} = 3\times 5 - 6\times 2 = 3$$

So, the inverse of the matrix is:

$$\begin{pmatrix} 3 & 6 \\ 2 & 5 \end{pmatrix}^{-1} = \frac{1}{3}\begin{pmatrix} 5 & -6 \\ -2 & 3 \end{pmatrix}$$

The solution is found as:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 5 & -6 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 21 \\ 16 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 9 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

This means that x=3 and y=2. If we insert these values in our equations, we will see that:

3x + 6y = 3 × 3 + 6 × 2 = 21
2x + 5y = 2 × 3 + 5 × 2 = 16

As we have discovered, there are a several functions in Excel dedicated to matrices. They all start with the prefix M. Here is the list: MDETERM, MINVERSE, MMULT and MUNIT.

They are all located in the Math & Trig group of functions. Some not related functions naturally start with the letter M, such as MOD or MROUND, but they are clearly not part of the matrix set of functions.

The additional function, that does not start with the letter M, is the function =TRANSPOSE(), which we also covered. The only one left uncovered from the above list is =MUNIT(dimension). It just returns the unit matrix for a specified dimension.

## 2.14 Eigenvectors, or eigenvalues.

Eigenvalue λ is the unique value (a scalar in the jargon of matrix algebra) that ensures that: |**A**-λ**I**|=0. In other words, if we multiply the unit matrix elements with this value λ, and subtract from the matrix **A**, the result must be zero.

Another way to think about eigenvalues is to think about them as the solutions to a set of equations. They are often called as characteristic roots or characteristic values, and sometimes also as the latent roots. Eigenvalues will form eigenvectors.

Remember what happens when you multiply a matrix with a vector. You get another vector. For example, **A v** = **X**. We'll take a look at an example.

Let's say that we would like to multiply a matrix **A** with vector **v**, as below:

$$\boldsymbol{A} = \begin{pmatrix} -1 & 2 \\ 4 & 6 \end{pmatrix} \times \ \boldsymbol{v} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \ \ \boldsymbol{Av} = \begin{pmatrix} 7 \\ 28 \end{pmatrix}$$

We already know how to do this. However, look at the result, it can be factored so the result could be:

$$\boldsymbol{Av} = 7\begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

Here we have a special case where we do not get **X**, but the same vector **v**, only this one is scaled. In this case: **A v** = λ**v**. We can see that λ is a scalar, and it is called an eigenvalue of the matrix **A**.

All square matrices have eigenvectors and their related eigenvalues. Not necessarily these eigenvectors have real numbers, i.e. they can have complex numbers. In these special cases when real eigenvectors and eigenvalues exist, it is the same if we multiply eigenvector by the matrix, or if we multiply the vector by a scalar.

The general principle is that **AX** = λ**X**. Where, **A** is a n × n matrix, and **X** is a vector. For example:

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \quad \boldsymbol{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

If we had an eigenvalue λ, then the corresponding eigenvectors satisfy the relationship:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Which is equivalent to:

$$\begin{pmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

A more compact way to write the above is: (**A**-λ**I**) **X** = 0, where **I** is the identity matrix. From there, eigenvalues are calculated. For a 2×2 matrix, for example, the eigenvalues are calculated as:

$$\lambda_{\pm} = \frac{1}{2}\left[(a_{11} + a_{11}) \pm \sqrt{4a_{12}a_{21} + (a_{11} - a_{22})^2}\right]$$

As we said, these eigenvalues form an eigenvector.

In general, det |**A**-λ**I**| = 0. This means that the starting point for finding the eigenvector will be the matrix **A** and the identity matrix **I**. Let's use a simple example:

$$\left|\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right| = 0$$

$$\left|\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right| = 0$$

$$\begin{vmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$$det \begin{vmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = (4-\lambda)(3-\lambda) - (2)(1) = 0$$

$$12 - 4\lambda - 3\lambda + \lambda^2 - 2 = 0$$

$$\lambda^2 - 7\lambda + 10 = 0$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad \lambda = \frac{7 \pm \sqrt{(-7)^2 - 4(1)(10)}}{2(1)}$$

$$\lambda_1 = 5 \quad \lambda_2 = 2$$

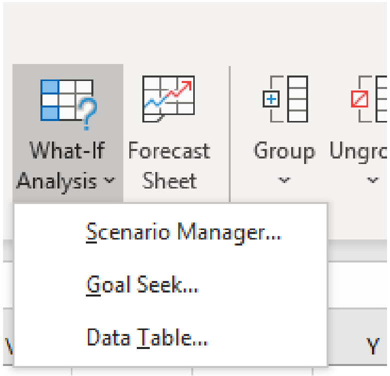
Using Excel, we might be able to make this process a bit more elegant and faster. Let's assume that we have matrix **A** in cells A3:C5. The identity matrix **I** is given in cells E3:G5.

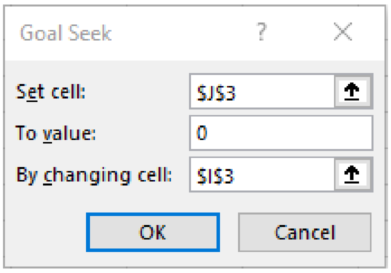
| | A | B | C | D | E | F | G | H | I | J | K |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | **Eigenvectors and eigenvalues** | | | | | | | | | | |
| 2 | Matrix A | | | | Matrix I | | | | Lambda λ | Det of A-Iλ | |
| 3 | 6 | 9 | 5 | | 1 | 0 | 0 | | 100.000 | -715275.0 | |
| 4 | 9 | 17 | 9 | | 0 | 1 | 0 | | | =MDETERM(A8:C10) | |
| 5 | 5 | 9 | 6 | | 0 | 0 | 1 | | | | |
| 6 | | | | | | | | | | | |
| 7 | Matrix A-Iλ | | | | | | | | | | |
| 8 | -94 | 9 | 5 | {=A3:C5-I3*E3:G5} | | | | | | | |
| 9 | 9 | -83 | 9 | | | | | | | | |
| 10 | 5 | 9 | -94 | | | | | | | | |

Cell I2 contains arbitrary number we selected (in this case 100) and it represents λ.

Cells A8:C10 represent a new matrix that was produced by subtracting from **A** the matrix **I** that has been multiplied by λ. This is the part of the equation |**A**-λ**I**| = 0. However, at present this does not amount to zero and in cell J3 we inserted the formula =MDTERM(A7:C9), which is the value of the determinant of A8:C10 and which we expect to be zero.

To calculate the value of λ, we need to invoke Goal Seek function in Excel. You can find it under the Data tab, and then click on What-If Analysis. If you click on this option, a dialogue box will appear as below:




What we are saying here is that we want cell J3 to be equal to zero and that this can be achieved by changing the values of cell I3. In other words, iterate through various values of λ, until the value of |**A**-λ**I**| is zero.

Excel quickly finds the first value of λ=27.07. See below how the cells changed after the Goal seeking operation.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Matrix A | | | | Matrix I | | | | Lambda λ | Det of A-Iλ | |
| 6 | 9 | 5 | | 1 | 0 | 0 | | 27.077 | 0.0 | |
| 9 | 17 | 9 | | 0 | 1 | 0 | | | =MDETERM(N8:P10) | |
| 5 | 9 | 6 | | 0 | 0 | 1 | | | | |
| | | | | | | | | | | |
| Matrix A-Iλ | | | | | | | | | | |
| -21.0767 | 9 | 5 | {=N3:P5-V3*R3:T5} | | | | | | | |
| 9 | -10.0767 | 9 | | | | | | | | |
| 5 | 9 | -21.0767 | | | | | | | | |

However, we know that eigenvector must have three eigenvalues, because the matrix **A** has 3 rows. To solve this problem, we used the same example, but appropriately expanded to three values of λ:

| | A | B | C | D | E | F | G | H | I | J | K | L |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 13 | Matrix A | | | | Matrix I | | | | Lambda λ | Det of A-Iλ | | |
| 14 | 6 | 9 | 5 | | 1 | 0 | 0 | | 0.923222 | 0.00 | =MDETERM(A19:C21) | |
| 15 | 9 | 17 | 9 | | 0 | 1 | 0 | | 1.000057 | 0.00 | =MDETERM(A24:C26) | |
| 16 | 5 | 9 | 6 | | 0 | 0 | 1 | | 27.0767 | 0.00 | =MDETERM(A29:C31) | |
| 17 | | | | | | | | | | | | |
| 18 | Matrix A-Iλ | | | | | | | | | | | |
| 19 | 5.076778 | 9 | 5 | {=A14:C16-I14*E14:G16} | | | | | | | | |
| 20 | 9 | 16.07678 | 9 | | | | | | | | | |
| 21 | 5 | 9 | 5.076778 | | | | | | | | | |
| 22 | | | | | | | | | | | | |
| 23 | Matrix A-Iλ | | | | | | | | | | | |
| 24 | 4.999943 | 9 | 5 | {=A14:C16-I15*E14:G16} | | | | | | | | |
| 25 | 9 | 15.99994 | 9 | | | | | | | | | |
| 26 | 5 | 9 | 4.999943 | | | | | | | | | |
| 27 | | | | | | | | | | | | |
| 28 | Matrix A-Iλ | | | | | | | | | | | |
| 29 | -21.0767 | 9 | 5 | {=A14:C16-I16*E14:G16} | | | | | | | | |
| 30 | 9 | -10.0767 | 9 | | | | | | | | | |
| 31 | 5 | 9 | -21.0767 | | | | | | | | | |

We created three λ cells I14:I16 as well as three |**A**-λ**I**| cells J14:J16. These cells use three different matrices, form A19:C21, A24:C26 and A29:C31. This effectively means that we have to run the Goal Seek routine three time, every time seeking the value in a different cell I14:I16. The three values that we get are: $\lambda_1$=27.07, $\lambda_2$=1 and $\lambda_3$=0.92.

It makes sense to put the initial value for λ as a relatively large number, that is by a magnitude larger than any number in the matrix. Once the first value of λ is found, we put for the next value of λ just a little bit smaller number. The third λ guess is just below the second optimal value of λ, which in the end gives us all three values of the eigenvector.

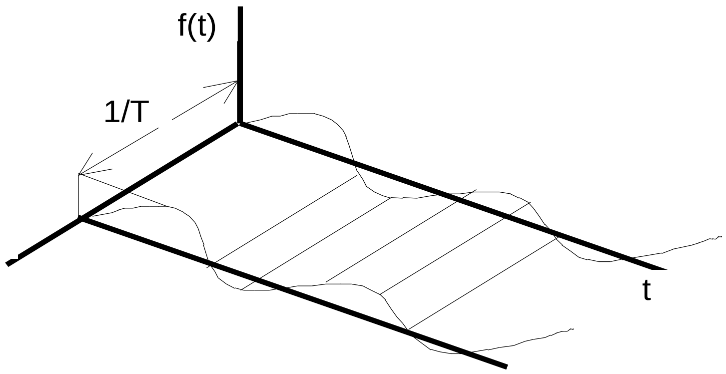
A little bit labour intensive, but certainly much quicker than manual calculations.

## 2.15 Frequency analysis and FFT

A periodic function is the one that repeats itself after a time 'T'. This is applicable if the function is presented in the so-called time space. However, the same function can be presented in a frequency space. Here is how to do it. We first define the frequency as:

$$f = \frac{1}{\mathrm{T}}$$

If we do this, the complex relationship between the time and the frequency space can be visualised in a three-dimensional graph as below.




To approximate such periodic functions we can use Fourier series (not shown here), which is a combination of sine and cosine terms. However, unlike the Fourier series, that can handle only periodic series, the Fourier transforms can handle any type of series.

If the time series is discrete, which most of the time series in business, economics and finance are, then the appropriate Fourier transform to be used is the Discrete Fourier Transform (DFT), which is defined by the following equation:

$$F_k = \sum_{n=0}^{N-1} x_n \, e^{\frac{-i2\pi kn}{N}}$$

Where, $x_n$ is the input signal, or time series, N is the total n umber of observations in the time series, k is a DFT coefficient corresponding to n, n is the number of DFT coefficients and *e* is Euler's number or the basis of the natural logarithms.

As the above equation contains imaginary number *i*, it is clear that $F_k$ (or DFTs) will be a series of complex numbers. A complex number of the form: z = a + b*i*, which as we know has a real and imaginary part.

The magnitude of the complex number is calculated as the modulus, which is $|z| = \sqrt{a^2 + b^2}$. This means that the magnitude of the complex number is equivalent to the amplitude of the DFT.

Once calculated, the series of DFTs, or $F_k$, becomes a series of complex number:

$$F_k = A_k + iB_k$$

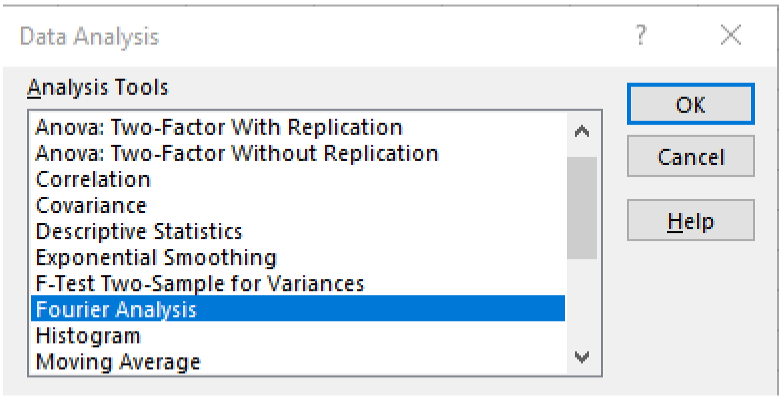
To calculate the amplitude, or magnitude, for $F_k$, we effectively need to use the formula for the modulus:

$$G_k = \sqrt{A_k^2 + B_k^2}$$

These amplitudes will show us how much is every frequency present in the time series of observations.

To calculate the coefficients $F_k$ can be extremely time consuming. For this reason, a short-cut method called Fast Fourier Transforms (FFT) was invented. Using this FFT method, Excel can also be used to calculate the Discrete Fourier Transforms (DFTs) or $F_k$.

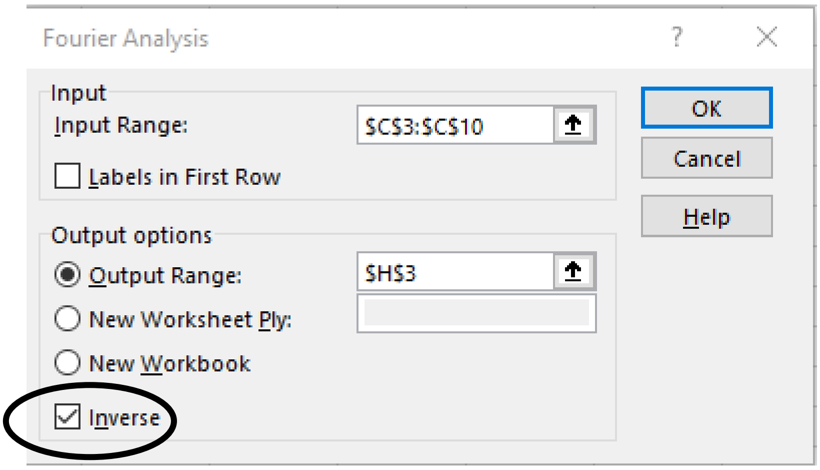
Excel Data Analysis Add-in routine for FFT is invoked from the Tools menu, Data Analysis option. Once this option has been selected, Data Analysis selection list, as below, appears. What follows is a self-explanatory wizard.



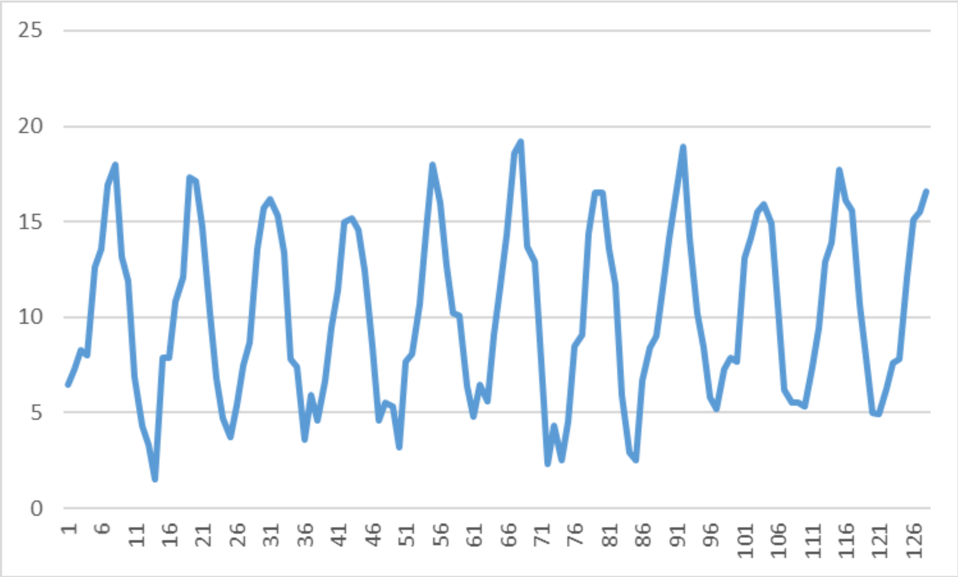

The result of applying this analysis is shown below. Column A contains a time series. The result is produced in column C, which is a transformation of column A, but given in a complex plane. These are effectively the DFTs, or a series of $F_k$ coefficients.

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | **Frequency analysis and FFT** | | | | | | | | | |
| 2 | **Time series** | | **Fourier transforms** | | | | | **Original series** | | |
| 3 | 0.536799 | | 3.91330499847005 | | | | | 0.536798864401462 | | |
| 4 | 1.019828 | | 1.16260751111195-0.380353337734352i | | | | | 1.01982762642818 | | |
| 5 | 0.553406 | | -8.2503737605037E-002-0.258760100288082i | | | | | 0.553405846748109 | | |
| 6 | 0.16638 | | -0.885364484196509-0.201689856642391i | | | | | 0.166380133502761 | | |
| 7 | 0.398177 | | -8.39266187916565E-003 | | | | | 0.398177350943741 | | |
| 8 | 0.089977 | | -0.885364484196509+0.201689856642393i | | | | | 8.99768388031646E-002 | | |
| 9 | 0.464074 | | -8.25037376050366E-002+0.258760100288082i | | | | | 0.46407410620213 | | |
| 10 | 0.684664 | | 1.16260751111195+0.380353337734351i | | | | | 0.684664231440501 | | |

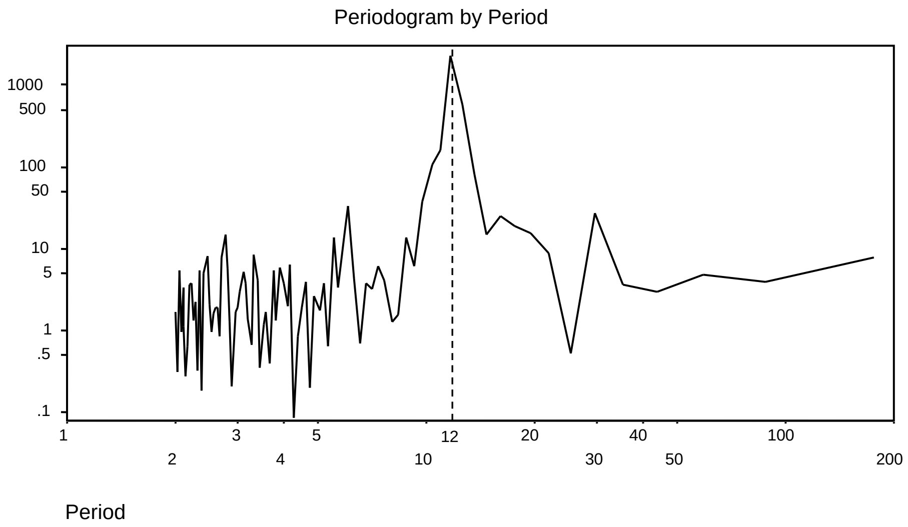
We said that Fourier transforms is just another way of representing the same data set. This means that we can revert the series of Fourier transforms back into the original time series from which they were created. We have done this in column H. The only difference is that in the dialogue box (see below) the input range is the Fourier transform series and we have to tick the Inverse box.



To illustrate better how to interpret these numbers, let's look at a graph below showing a typical seasonal time series plot:

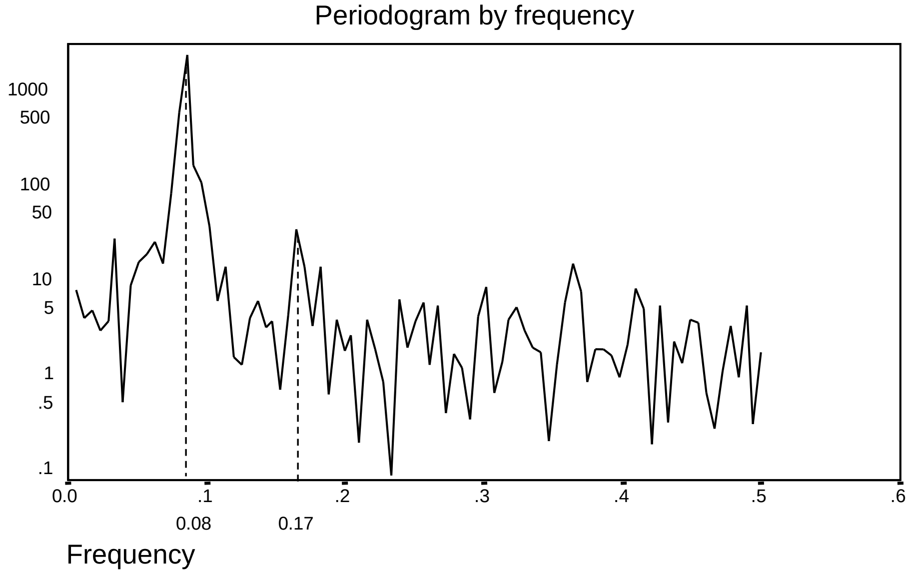



The above series is presented in the time domain, which means that every observation is recorded chronologically on the 'x' axis. If the series, for example, constitutes monthly data, then an observation for one month represents 1/12 of the year, although not necessarily a proportional part i.e. not all the values for all the months are the same. If we translate the series from a time domain into a frequency domain, the same series will look like below.




The 'x' axis consists of periods and in this case, it is chosen to be logarithmic (as is the y-axis). This graphical presentation is called a periodogram.

However, we can change periods into frequencies (see the graph below), which is yet another way of representing the same series.




The word cycle usually implies that something will go up and down and will repeat itself after a period of time. If our time series of length ‘n’ consists of the shortest possible cycle (one up, one down observation interchangeably), then the maximum number of cycles for this frequency is n/2. This means that the number of cycles can reach maximum for 50% of the number of observations. If, on the other hand, the whole series consists of just one cycle, then the frequency is only 1/n. The lowest frequency possible has zero cycles, which means there is no cycle at all. Frequencies are, therefore, calculated as the number of times the cycle repeats itself in a series over the total number of observations.

An alternative way of looking at this is to say that if we have monthly data, then one period constitutes 12 months. The frequency of this series is 1/12 cycles per month. This effectively tells us what we have already stated, that the frequencies and periods are reciprocals of one another. Because the peak in the graph in periodogram shows number 12, this indicates that the periodicity of the series is 12 months. On the other hand, the graph in the frequency chart shows the highest frequency to be 0.08 (which is in fact 1/12), confirming that these two graphs show one and the same thing, just from a different point of view. By the way, looking at other frequencies on the frequency graph, we can see that the second highest frequency is period six, which is equivalent to 0.17 (i.e.1/6) from the periodogram.

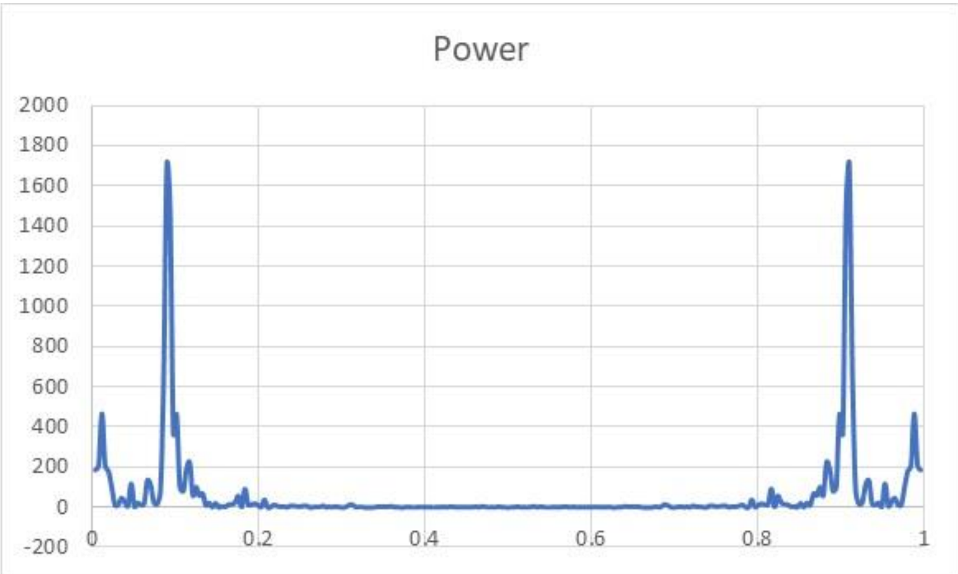
If the amplitudes $G_k$ for every DFT coefficient $F_k$ are normalized, then the graph that shows all the frequencies against their amplitude (or the power) is called the Power Spectrum.

We will use an example and combine Excel Data Analysis FFT routine to calculate the power spectrum for the average annual sunspot number from 1761 until 2016. The sheet below shows the first six years and the last seven years of that interval.

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 1 | Average annual | 256 | k=Number of DFT coefficients (k=1,2,..,N-1) --> Has to be $2^n$ | | | 0.5 | Folding frequency | |
| 2 | sunspot example | 256 | N=Number of observations | | | 0.003906 | Frequency increments | |
| 3 | | 256 | T=Total time (number of years in this case) | | | | | |
| 4 | **Year** | **Avrg sunspot** | **DFT** | **Magnitude** | **Frequency** | **Power** | | |
| 5 | 1761 | 143.2 | 21314.1 | 166.51641 | 0 | 13863.86 | | |
| 6 | 1762 | 102 | 980.741382421425+1444.43907199972i | 13.640048 | 0.0039063 | 186.0509 | | |
| 7 | 1763 | 75.2 | -488.014150390877+1779.88448838923i | 14.418554 | 0.0078125 | 207.8947 | | |
| 8 | 1764 | 60.7 | 1276.61982356039-2449.93097274913i | 21.582758 | 0.0117188 | 465.8154 | | |
| 9 | 1765 | 34.8 | -903.554433703613-1620.4011345227i | 14.494473 | 0.015625 | 210.0897 | | |
| 10 | 1766 | 19 | -1715.59132676931-136.905024266534i | 13.445666 | 0.0195313 | 180.7859 | | |
| 254 | 2010 | 24.9 | 136.845294389014+425.06977732873i | 3.4887073 | 0.9726563 | 12.17108 | | |
| 255 | 2011 | 80.8 | -1295.39839487652+11.9138925316954i | 10.120728 | 0.9765625 | 102.4291 | | |
| 256 | 2012 | 84.5 | -1715.59132676931+136.905024266545i | 13.445666 | 0.9804688 | 180.7859 | | |
| 257 | 2013 | 94 | -903.554433703606+1620.40113452271i | 14.494473 | 0.984375 | 210.0897 | | |
| 258 | 2014 | 113.3 | 1276.61982356041+2449.93097274913i | 21.582758 | 0.9882813 | 465.8154 | | |
| 259 | 2015 | 69.8 | -488.014150390886-1779.88448838923i | 14.418554 | 0.9921875 | 207.8947 | | |
| 260 | 2016 | 39.8 | 980.741382421415-1444.43907199972i | 13.640048 | 0.9960938 | 186.0509 | | |

We happen to have 256 observations (cell B2 shows this) and we will calculate the full 256 DFT coefficients (cell B1). In practise, the time series can be any length and it is customary to calculate $2^n$ number of DFTs, which is 64, 128, 256, 512, 1024, or any other number of frequencies, depending how fine is desired resolution.

The other two cells that need explaining in the sheet above are F1 and F2. F1 contains the folding frequency. The power spectrum turns into a mirror image of itself after the folding frequency (see the picture below).




For this reason, it does not make sense to calculate amplitudes beyond the folding frequency. Folding frequency $f_f$ is calculated as:

$$f_f = \frac{T}{2N}$$

Where T is the time and N is the number of observations. In our case, the folding frequency in cell F1 is 0.5.

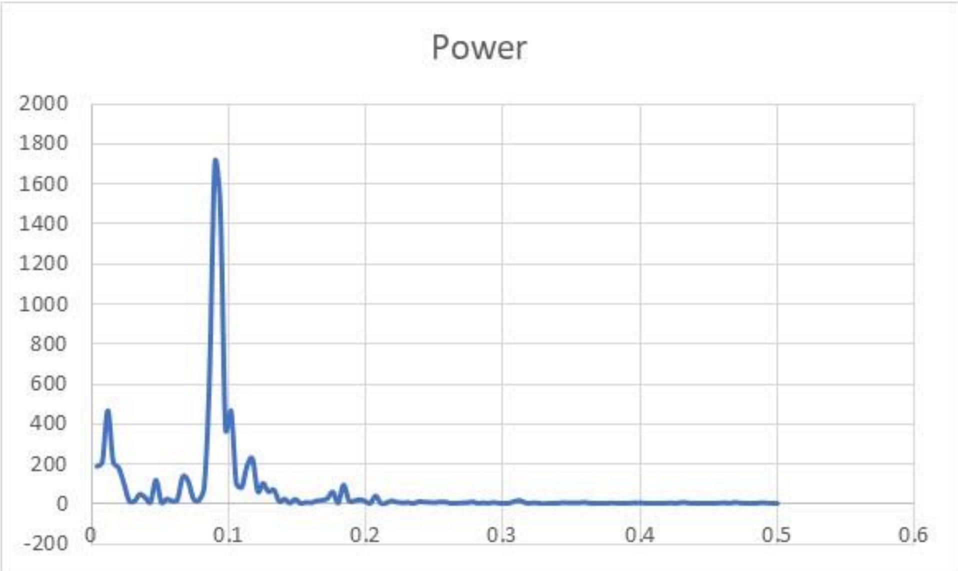
This folding frequency will also help with calculating the frequency increments $f_c$. The equation is:

$$f_c = \frac{2f_f}{T}$$

In our case the frequency increment is 0.003906, as in cell F2. This means that we will start with frequency 0, add to it the first increment, then add to this sum the next, etc. Column E in the above sheet shows these frequency increments. We will continue to do this until we reach $f_f$. After that, it makes no sense to calculate further coefficients, as we already said.

Column C in the above sheet is produced by using Excel Data Analysis FFT Add-In, as shown before. This column consists of complex numbers. From this column, we isolate the amplitudes (or the magnitudes for every frequency) in column D. This is achieved using Excel formula =2*IMABS(C5)/$B$1. The function =IMABS() produces a modulus of a complex number from column C, which has been normalized by the number of coefficients.

And finally, in column E we calculate the power for every frequency as the square value of column D. The result is the graph as below. We eliminated from the graph the first frequency ($f_0$), which is always disproportionally high, in order to see better the rest of the power spectrum.




The highest amplitude (or magnitude) is 1706.9 for the frequency of 0.089. If you calculate 1/0.089, you get 11.13, which means you are converting the frequencies back into the original time measure. As our time series shows annual average sunspots, then 11.12 means that the sunspots show periodicity of 11.13 years, which is exactly what astronomers are telling us.

## Workbook summary

The objective of this chapter was primarily to provide elementary refresher into some of the most often used mathematical techniques that student might need to follow this textbook with confidence and without effort.

The secondary objective was to provide a brief introduction and/or a refresher into more advanced mathematical concepts. This was achieved by relying on Excel as the key platform to simplify the calculations and explain the material in a more concise way.

## Want to learn more?

The textbook online resource centre contains additional spreadsheet material that accompanies this chapter.